COURSE GUIDEBOOK

The Joy of Mathematics

Part I

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Course No. 155

The Joy of Mathematics, Part I
Professor Murray H. Siegel

COURSE GUIDEBOOK

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Part I

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Sam Houston State University

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Sam Houston State University

Although Dr. Siegel is known nationally as a mathematics leader in our public schools, much of his professional life has been devoted to adult education. His community workshops, college courses, college workshops, and videos have one purpose. The primary objective is to help adults overcome mathematical anxiety and to provide his audiences with a picture of mathematics as a subject with logical underpinnings and great utility. In addition, he tries to focus on the connectivity of the various branches of mathematics, as well as the beauty that exists throughout the subject.

Murray Siegel was born and raised in Brooklyn, NY. He was educated in New York City schools, where he met two of his three "Great Teachers." Mary Doyle (his sixth-grade teacher) and Sally Woroner (his ninth-grade teacher) inspired him with the knowledge that he had no limits other than those he placed on himself and that there was no excuse for accepting anything less than excellence. He received a B.S. in Physics from New York University College of Engineering. At his alma mater he met his third "Great Teacher," Harry Park, who allowed him to look more clearly at himself so that he could know others better. Dr. Siegel accomplished his graduate studies in mathematics education at Georgia State University, where he received his M.Ed. Ed.S. and Ph.D. Dr. Siegel is currently Assistant Professor of Mathematics in the Department of Mathematical and Information Sciences at Sam Houston State University, Huntsville, TX.

After a stint in the USAF and in business, Dr. Siegel decided that he really belonged in teaching. Of course, his engineering, military, and business experience has allowed him to see mathematics as a tool. He has attempted to pass this view along to all his students, adults as well as children. Dr. Siegel claims that teaching mathematics is a missionary vocation. This tape series is truly an attempt to bring the meaning of mathematics to many.

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Part I

The Joy of Mathematics

Scope:

This twenty-four lecture series has seven general topics, plus a concluding lecture that should bring closure to your experience. The general topics are: numbers, algebra, calculus, fractals, geometry, probability, and data analysis.

Part I

Numbers

The first five lectures of the series attempt to trace the development of the use of numbers by humans from the Egyptian hieroglyphics to our current number system. The various types of numbers we use, including whole numbers, integers, rational numbers, irrational numbers, and complex numbers, are investigated. The sense that the rules of numbers are arbitrary or "magical" should be replaced with a sense of understanding about *why* we do what we do and *who* did it first.

Algebra

Lectures Six through Nine are devoted to the demystification of algebra. Algebra is shown to be a useful tool in understanding arithmetic. A significant amount of time is spent investigating the use of algebraic functions to model situations in our world. A linear function is used to explore the relationship between fat and calories in pizza. A quadratic equation models the trend in the population density of the United States. A cubic function provides a visual image of the growth of the number of high school soccer players. The acceleration of federal expenditures on social insurance is diagnosed with an exponential function. The utility of algebra is exposed so that no student should ever ask, "When am I going to use algebra?"

Calculus

Two lectures provide a brief acquaintance with calculus. The seventeenth century was a wonderful time for new ideas. Kepler had produced mathematical models for planetary motion but there was no mathematics available to deal with a physical measure that had velocity and acceleration. Working independently, Newton and Leibniz produced calculus. Some years later, Taylor and Maclaurin used calculus to create power series. Lectures Ten and Eleven look at the derivative and the integral, followed by a view of power series seen through the use of a graphing calculator.

Fractals

Lecture Twelve is devoted to the beauty of fractals. Investigations into this branch of mathematics started at the end of the nineteenth century. Lack of computer power kept the investigations restricted to the abstract. In the last part of the twentieth century, we have been awed by the complexity of the fractal

image. But fractals are not simply art; they are being used to allow scientists to understand chaotic phenomena such as hurricanes. Part I of the course ends with Lecture Twelve.

Part II

Geometry

We start Part II of our exploration into the joys of math by going back into history and attempting to appreciate the great geometric thinkers of the ancient world (Lecture Thirteen). Thales, Pythagoras, Euclid, and Apollonius built a foundation of understanding that we use today. The patterns of geometric relationships are discovered and the rationale for all those geometric formulas is brought to light. René Descartes took his love of geometry and applied it to algebraic problems and created analytic geometry. The graph paper we use to provide models for the abstract was his invention. Did you ever wonder where they obtained the names for the trigonometric functions—sine, tangent, and secant? Actually, one of the names was created by a mistaken translation, but the other names make a great deal of sense. Lecture Fifteen will discuss triangles and trigonometry, and Lecture Sixteen covers conic sections.

Probability

We live in an uncertain world. Insurance is based on uncertainty and many people view investment as a form of gambling. Probability is the mathematical analysis of this phenomenon. It turns out that gambling played a significant role in the early development of the mathematics of probability. This section will investigate the binomial and normal distributions and how we can apply them in problem-solving situations. The use of simulations to provide data when none are available is demonstrated. After viewing these three lectures on probability (Lectures Seventeen through Nineteen), you should have a better understanding of the meaning of "the chance of rain tomorrow is thirty percent."

Data Analysis

Four lectures involve the treatment of data. Analyzing a set of data, comparing sets of data, and seeking a statistical model for a relationship between two variables are discussed. The use of sample results to make an inference about the population is the focus of the last two lectures on data analysis. The use of the hypothesis test to determine a level of significance and the use of a confidence interval to estimate a population parameter are both explained using realistic examples. How do they know how many people support a particular political philosophy?

The final lecture starts with the question, "What would mathematics be if our ancestors had only had eight fingers rather than ten?" A new perception of mathematics is the primary objective of this series. By Lecture Twenty-Four, the walk through history, the breaking down of mathematical secrets, the use of

pattern recognition, and the exposure to the beauty of mathematics should allow the viewer to have that new (and hopefully more positive) view of the subject.

Note:	Dr. Siegel has made two other mathematics courses for The Teaching Company. These would be appropriate for students who would like a review of basic mathematical concepts and operations. The courses are *Basic Math* and *Algebra II*. Additionally, The Teaching Company has two other high-school math courses: *Geometry* by James Noggle and *Algebra I* by Dr. Monica Neagoy, both of which cover topics discussed in *The Joy of Mathematics.*

Lecture One

An Introduction and the Counting Numbers

Scope: This lecture offers an overview of the entire series. This series will cover topics in numbers, algebra, calculus, fractals, geometry, probability, and statistics, with emphasis placed on the developmental history and application of the various topics. As an example, I will relate my own personal experience with the uses of mathematics in various arenas. Our investigation of numbers begins with the counting numbers. The fascinating development of numerals is traced from ancient times to the present. The lecture will also cover the classification of counting numbers and a brief investigation of patterns in mathematics exemplified by modulo number systems. Finally, I will introduce numerology and the human fascination with specific numbers.

Outline

I. Mathematics can be considered as a beautiful tapestry.

A. A weaving of various perceptions of mathematics:

1. The history of the development of mathematics traces the cognitive development of humans as they first searched for an understanding of their world and then sought to use their brains to control the world.
2. There are connections between mathematics and our world that we often fail to realize.
3. The patterns inherent in mathematics, when understood, make us better problem solvers, because pattern recognition is a key tool in recognizing and solving a problem.
4. We use mathematics all the time, and this series will analyze the various applications of mathematics in many areas of human endeavor.

B. The mathematics in this series is subdivided into seven primary areas or subsets.

1. Numbers are the basis of mathematics, from the fundamental counting numbers to the abstraction of the complex set.
2. Algebra seeks to generalize arithmetic through the use of symbols, thus providing a language for science, business, and other areas.
3. Calculus is the study of change and the effects of change. It was made necessary by our investigation of the heavens and has become a useful tool in many facets of human effort.
4. Fractals is a relatively new area of mathematics that is being used to develop models for seemingly chaotic behavior. Fractal geometry has great artistic beauty and serves as a vehicle for scientific investigation.
5. Geometry, which means to measure the earth, goes back to the tax collectors of the Egyptian pharaoh. Geometry teaches us precision and logical development of ideas.
6. Probability started as a study of the outcomes of gambling situations but has become a scientific way to deal with the chance variation of life.
7. Statistics is really a social science that is an intense user of mathematics. Collecting, displaying, and analyzing data have become facts of life, from political polls to statistical process control in manufacturing.

II. Who is Dr. Murray Siegel?

A. I am a teacher of mathematics whose professional life has been devoted to convincing people that mathematics can be easily understood and is a vital part of our lives.

1. I have been a public school teacher for 25 years, having taught mathematics to young children, adolescents, and adults.
2. I have been, and continue to be, an adjunct faculty member at numerous universities and community colleges.
3. I have taught mathematics to adults in various media—the classroom, the workshop, and through videotapes such as this series.

B. I have used mathematics in the various areas of my work experience.

1. As an engineering student and as an assistant to the plant engineer, I became aware of the important applications of the mathematics that I had learned in school.
2. I served as a flying officer in the USAF and used mathematics in navigation and electronic warfare.
3. Before becoming an educator, I was a manager and an executive in the business world. This experience included positions in the investment industry, where the importance of mathematics is obvious.
4. As a student of mathematics, I have used my knowledge to enhance my effectiveness as a consumer and an investor.
5. I honestly believe that mathematics is not a subject for a select elite. If you lack confidence in mathematics, I hope this series will convince you that the subject can be understood and appreciated by all.

III. The first numbers created by humans were the counting numbers.

A. They are sometimes called the natural numbers.

1. The earliest humans counted and used fingers to communicate counts.

2. They used marks scratched on a rock or a wall to record counts.
3. Eventually sounds were associated with various numbers.
4. Some examples would include counting clan members, the number of animals killed for food, the number of bowls owned by the clan, the number of "moons" since the last snow fall, etc.

B. Eventually symbols had to be invented to be able to write bigger numbers.
1. The Egyptians developed hieroglyphics to symbolize certain basic numbers. The earliest use of hieroglyphics to represent numerical quantities in Egypt dates to 3400 B.C.
2. A rod represented 1. An arch (sometimes interpreted as the heel of a hand) represented 10. A scroll or coil represented 100. A lotus flower represented 1000.
3. Eventually the Egyptians created symbols for 10,000; 100,000; and 1,000,000.
4. The hieroglyphic for 1,000,000 was supposed to represent an astonished man.

C. Some cultures used alphabet letters for numerals.
1. The Greeks used letters for numbers as early as 600 B.C.
2. The first ten letters were the numerals for 1 through 10.
3. The eleventh letter, kappa, represented 20.
4. The next seven letters represented 30, 40, 50, 60, 70, 80, and 90.
5. Rho was 100, sigma was 200, tau was 300, etc.
6. The Hebrews and Phoenicians had similar alphabets and used the letters to represent numbers in a way similar to what the Greeks had started in the fifth century B.C.

D. The Romans used letters from their alphabet but did not use successive letters as the Greeks and Hebrews did.
1. Little is understood about the earliest development of the Roman numerals.
2. By 500 B.C., the Romans had overcome the Etruscans and early Roman monuments have inscriptions using Roman numerals.
3. I represented one (a single finger).
4. V represented 5 (one hand).
5. X represented 10 (two hands).
6. The original Roman symbol for 50 was an arrow facing downward, which eventually became an L.
7. C, for the Latin word *centum*, represented 100.
8. M, for the Latin word *mille*, represented 1000.
9. The D, which was used for 500, was actually one half of the symbol used for 1000 before M became the common symbol.

E. The Romans used addition and multiplication to write numbers.
1. IIII was 4.

2. DCCCLXII represents 862.
3. Until the Middle Ages, VM did not mean 5 less than 1000.
4. VM would represent 5000 (5 times 1000).
5. Once the printing press with movable type was invented, the use of subtraction became common in writing Roman numerals.
6. IX replaced VIIII as 9.

IV. A number system that was based on place value was needed to make computation easier and to allow for writing very large numbers.

A. The Hindus of India developed such a system. Little is known about the influence of the Greek, Babylonian, and Chinese number system on the evolution of the Hindu system.
1. Arab armies invaded India in their effort to spread Islam.
2. The Arabs recognized the value of this number system and brought the idea back to Baghdad, where it became the Arab system.

B. In 825, the numeral forms and the concept of place value had become accepted by the Arabs but credit was always given to the Hindus for developing the system.

C. The Hindu-Arabic system spread across North Africa to Muslim Spain, but it was not accepted in Europe because it had been created by non-Christians.

D. In the thirteenth century, Leonardo of Pisa (or Fibonnaci) used the Hindu-Arabic system in his writings, which convinced European scholars to see the benefit of this system over the use of Roman numerals.

E. By the fifteenth century, the invention of the printing press caused the forms of the ten numerals to become standardized.

F. The Hindu-Arabic system did not replace Roman numerals in some European schools until the end of the sixteenth century. European bookkeepers did not uniformly use Hindu-Arabic numerals until about 1700.

V. Modulo number systems can be used to develop an understanding of our number system.

A. A modulo five system uses only five numbers—0, 1, 2, 3, and 4.

B. A modulo system can be modeled by a "clock" and is sometimes called "clock arithmetic."
1. In mod 5, $4 + 3 = 2$, because 7 is 2 more than 5.
2. In the same way, $4 \times 4 = 1$, because 16 is 1 more than 15, which is 3 times the modulus.
3. A look at the multiplication table for mod 5 reveals that each line has all five numbers (0, 1, 2, 3, and 4).
4. The mod 6 multiplication table has only two lines that contain all six numbers—one and five.

5. Now we are seeking mathematical truth based on our attempt at pattern recognition.
6. A possible first thought is that if the modulus is odd, each line contains all numbers but if the modulus is even, each line will not contain all the numbers.
7. Looking at the mod 9 multiplication table contradicts that hypothesis.
8. The following modulo systems have all numbers on all lines: 2, 3, 5, 7, 11, 13.
9. Do you see the pattern?

C. Modulo systems with a prime number for a modulus will exhibit the characteristic.
   1. Systems with a composite modulus will have the characteristic only for numbers that are relatively prime with respect to the modulus.
   2. What are prime numbers and how were they discovered?

VI. Numbers took on meaning as people searched for the meaning of life.

A. The ancient Egyptians and Greeks believed that numbers held the key to understanding the world.
   1. Eratosthenes, a Greek who lived in Egypt in the third century B.C., developed a sieve that was used to identify prime numbers.
   2. A prime number can only be divided by 1 and itself.
   3. The number 1 was not considered to be prime; it was the creator number from which all numbers were created.
   4. Numbers that were multiples of prime numbers were crossed out in the sieve and the remaining numbers were prime.
   5. It was realized that the only even prime number was 2.
   6. Odd numbers, even if they were not prime, were given a higher status than even numbers.
   7. Male children were given names with an odd number of letters, while female babies were given names with an even number of letters.
   8. Even today, people will tend to pick the numbers 3 or 7 if asked to select a number from 1 through 10.

B. Because Greek and Hebrew used letters for numerals, numbers that "spelled" words had magic meaning.
   1. The Greek letter alpha became a religious symbol, because 1 was the creator number.
   2. In Hebrew, the number 18 spells the word "chai," which means "life," so even today many Jewish people make charitable donations in multiples of 18.
   3. The Greek spelling of Emperor Nero's name adds up to 666, which might explain that number being used as the "sign of the beast" in the Book of Revelations.

Essential Reading:

Georges Ifrah, *From One to Zero*.

NCTM, *Historical Topics for the Mathematics Classroom*, Section II.

Supplementary Reading:

Howard Eves, *Great Moments in Mathematics (Before 1650)*, Lecture 1.

Jan Gullberg, *Mathematics—From the Birth of Numbers*, Chapters 1 and 2.

Questions to Consider:

1. Try dividing 2314 by 56 using Roman numerals and try to envision how a thirteenth-century European could justify using Roman numerals in lieu of the Hindu-Arabic system.
2. Develop a set of rules for adding and multiplying odd and even numbers. For example: An odd times an odd is always ___, or adding an odd number of even numbers always provides an ___ answer.
3. Write a list of all the ways you personally use mathematics in your daily life. Write a second list of all the items you use that involve mathematics. What can you conclude from your lists?

Lecture Two

Patterns with Counting Numbers

Scope: This lecture investigates the use of counting numbers in various patterns. Some of the patterns are quite famous in history. The triangular numbers, the Fibonacci sequence, and Pascal's triangle are examples of some famous patterns. Other patterns are not as widely known but they offer a fascinating look at the power of mathematics. Perhaps of greater importance, our investigation of these patterns will begin to provide an insight into what a mathematician actually does. Palindromes, "black holes," the sum of cubes, and Hank Aaron numbers are examples of patterns that are not as well known as those previously mentioned. This lecture mixes a bit of history, some mathematical investigation, and a good bit of fun.

Outline

I. Palindromes are words or numbers that are the same forwards or backwards.

A. There are many examples of palindrome words and phrases.

1. Names such as Anna, Otto, Mom, Dad, and Bub are palindromes.
2. Radar and racecar are examples of palindrome words.
3. A classic palindrome sentence is: "A man, a plan, a canal, Panama."

B. Examples of palindrome numbers would include 33, 414, 8778, and 10401.

1. The year 1991 was a palindrome year and the next palindrome year will be 2002. These years are separated by only eleven years.
2. The closest palindrome year previous to 1991 was 1881, which was 110 years before 1991.
3. The closest palindrome year after 2002 is 2112, which is 110 years after 2002.
4. The change of millennia offers a rare opportunity to experience two palindrome years in a lifetime.

C. To determine if a number is a palindrome, reverse the order of the digits. If this number is the same as the original, then you have a palindrome.

1. Consider 142. If you reverse 142 you get 241, so 142 (and 241) is not a palindrome.
2. If you add 142 + 241 the sum is 383, which is a palindrome.
3. A mathematician would ask, "Do you always get a palindrome?"
4. If you start with 251 and you reverse-and-add, the sum is 403. If the reverse of 403, 304, is added to 403, the sum is 707, which is a palindrome.

D. The question that must be answered is: Will all numbers under the reverse-and-add process always become a palindrome?

1. If we consider the two-digit numbers, the multiples of eleven are all palindromes.
2. Some of the two-digit numbers, such as 18 and 34, become palindromes after one reverse-and-add cycle.
3. Other two digit numbers, such as 57, take more than one cycle, and 89 takes more than twenty reverse-and-add cycles to finally obtain a palindrome.
4. It is conjectured that all counting numbers will eventually become palindromes under the reverse-and-add technique, even though for some numbers it takes more than thirty cycles to obtain the palindrome. This conjecture has not been proven for all counting numbers.
5. One possible use for this kind of pattern analysis is in creating and deciphering codes.

II. A black hole in physics is a location where the matter is so dense that the intense gravity prohibits anything, including light, from escaping.

A. Start with the number 6174 and write the largest and smallest four-digit numbers that can be written using the four digits in 6174. Finally, subtract the smaller number from the larger.

1. Subtracting 7641 - 1467, the difference is 6174. The answer is the number with which we began the problem. We are in a black hole, because there is no escape in using this process.
2. Start with 3870 and complete the same process. The largest number is 8730 and the smallest is 0378. The difference obtained from 8730 - 0378 is 8352.
3. Is 8352 a black hole number? 8532 - 2358 is 6174.
4. If you start with any four-digit number other than one where all the digits are the same and you apply this subtraction process, you will always wind up at 6174, which is a black hole.

B. What happens if you use a three-digit number?

1. Starting with a three-digit number where all the digits are not the same, if the same subtraction process is applied, you will eventually get "stuck" at 495, which is the three-digit black hole number.
2. It is interesting to note that both 6174 and 495 are multiples of 9.
3. By examining some examples of the process with three-digit numbers, it becomes apparent why you always wind up at 495.
4. For any whole number, no matter how many digits it has, this process will always terminate at a black hole and the black hole number will be a multiple of 9.

III. Another interesting pattern involving counting numbers uses the sum of the cubes of the digits.

A. The cube, or third power, of each of the ten digits is easy to find. Zero cubed is 0, 1 cubed is 1, 2 cubed is 8, 3 cubed is 27, 4 cubed is 64, 5 cubed is 125, 6 cubed is 216, 7 cubed is 343, 8 cubed is 512, and 9 cubed is 729.

1. If we start with 43 and sum the cubes of the two digits, we get 64 + 27 = 91.
2. Summing the cubes of the digits of 91 gives 729 + 1 = 730.
3. Continuing the process, we get 343 + 27 + 0 = 370.
4. We note that the digits in 370 are the same as the digits in 730. When we sum the cubes of the digits of 370 we get 370.

B. If we start with 207 (a three-digit number), the sum of the digits is 8 + 0 + 343 = 351.

1. Continuing the process, the sum of the cubes of the digits of 351 is 27 + 125 + 1 = 153.
2. Because 153 and 351 are comprised of the same three digits, we now have a lock.
3. When we started with 43, we locked at 370, and when we stared with 207, we locked at 153.
4. Does the number of digits dictate the lock number?
5. If we start with 16, the sum of the cubes is 1 + 216 = 217.
6. The sum of the cubes of the digits of 217 is 8 + 1 + 343 = 352.
7. Continuing the process, the sum of the cubes of 352 is 27 + 125 + 8 = 160.
8. We note that 160 will give us the same sum of cubes as the number with which we began this process, 16. Instead of terminating in a lock, the process ends in a three-number loop, 160 - 217 - 352.

C. When the sum-of-cubes process is applied to a counting number, the process will terminate in one of nine ways.

1. There are five numerical locks: 1, 153, 370, 371, and 407.
2. There are two types of endings that are two-number loops: 136 - 244 and 919 - 1459
3. There are two types of endings that are three-number loops: 160 - 217 - 352 and 250 - 133 - 55.
4. If you try this process with the nine three-digit counting numbers in which all the digits are the same (111, 222, etc.), the sum of the cubes locks at 153.
5. If you start with the nine two-digit counting numbers in which both the digits are the same, only 33, 66, and 99 lead to 153.
6. What is so special about these numbers? All are multiples of 3, and all multiples of 3 terminate the sum-of-cubes process with 153.

IV. Patterns and sequences of counting numbers abound. Some are quite useful; others must simply be enjoyed for the inherent beauty of the pattern.

A. There are "magic" sequences of numbers, such as 7, 11, and 13.

1. Select any three-digit number such as 368 and multiply this number by 7. Then multiply the result by 11 and multiply that product by 13.
2. The final product is 368,368. We have created a six-digit number with the original number repeated. This will work for any three-digit number.
3. Take a six-digit number in which the first three digits are the same as the last three digits (such as 925,925) and divide this number by 7. Take the result and divide it by 11, then divide that quotient by 13.
4. The final result is a three-digit number that is the same three digits that were in the original number (925). Why does the sequence 7, 11, 13 have "magical" powers?
5. It really is not magic. The product of 7, 11, and 13 is 1001. Multiplying a three-digit number by 1001 creates the six-digit number with the repeating cycle of three digits.
6. Dividing a six-digit number with a three-digit cycle by 1001 results in a quotient equal to the three-digit cycle.

B. In April 1974, Henry "Hammerin' Hank" Aaron hit his 715th major-league home run. This career total made him the all-time leader in baseball.

1. The player whose total he passed was, of course, George Herman "Babe" Ruth, who had hit 714 major-league home runs during his career.
2. A mathematician who worked at the University of Georgia and who was a baseball fan saw a pattern that involved 714 and 715. If you multiply the two numbers, the product is 510,510.
3. Because this number is a six-digit number with a three-digit cycle, we know that 7, 11, and 13 are factors.
4. Because the number ends in 0, we know that 2 and 5 are factors. The six digits add up to 12, which means that 510,510 is divisible by 3.
5. The final factor is 17. We can see that 2 x 3 x 5 x 7 x 11 x 13 x 17 = 510,510.
6. Thus 714 x 715 is equal to the product of the first seven prime numbers. No prime from 2 through 17 is missing.
7. Two adjacent numbers whose product is equal to the product of consecutive primes are called "Hank Aaron" numbers. Five and six are Hank Aaron numbers, because 5 x 6 = 30 and 30 is the product of the first three primes (2 x 3 x 5).

C. Leonardo of Pisa came up with a sequence of numbers that has become known throughout the world as the Fibonacci numbers.
   1. We start with two 1s. The sequence progresses by obtaining the next term by adding the two previous terms.
   2. The third term is 2 since 1 + 1 = 2. The first three terms are 1, 1, 2.
   3. The fourth term is 3 since 2 + 1 = 3. The sequence is 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, …
   4. The Fibonacci sequence has applications in biology and geometry and, of course, can be used to create codes.

V. There are two sequences that are associated with triangles: They are the triangular numbers and Pascal's triangle.
   A. Triangular numbers start with 1 and then add 2, add 3, etc.
      1. The first ten triangular numbers are 1, 3, 6, 10, 15, 21, 28, 36, 45, 55.
      2. This sequence is called the triangular numbers, because each number can be formed with a triangular arrangement of dots. For example 6 would have three dots on the bottom row, two dots on the second row, and one dot on top.
      3. By adding four dots to the bottom of this arrangement, you get the triangular number 10.
      4. Triangular numbers are used in geometry. The number of lines that can be formed using a specific number of points is a triangular number. The total of the number of sides of a polygon and its diagonals is a triangular number.
      5. If you add two adjacent triangular numbers, you get a square number. For example 6 + 10 is 16, which is 4 squared.
   B. Blaise Pascal, a seventeenth-century French mathematician and philosopher, is given credit for creating this triangle. Actually the arrangement of numbers in Pascal's triangle was known hundreds of years before Pascal's birth.
      1. The first row of the triangle has a single 1.
      2. The second row has two 1s. Each row starts and ends with 1.
      3. Each number between the 1s is the sum of the two numbers above it. The third row is 1 - 2 - 1. The 2 is the sum of the two 1s in the second row.
      4. The fourth row is 1 - 3 - 3 - 1 and the fifth row is 1 - 4 - 6 - 4 - 1.
      5. The rows of Pascal's triangle can go on forever. There are many patterns embedded in Pascal's triangle. The most obvious are the counting numbers and the triangular numbers.
      6. The sum of each row is a power of 2. The first row is the 0 power of 2, 1.
      7. The sum of the third row is 4, which is 2 to the second power.
      8. The rows of Pascal's triangle also display the powers of 11. The first row is 1, which is the 0 power of 11. The second row is 11, which is the first power of 11. The third row is 121, which is 11 to the second power, and the fourth row is 1331, which is 11 to the third power.
      9. Pascal's triangle is used in probability to determine the number of ways a particular event (e.g., the flipping of a coin) can occur. Most people admire Pascal's triangle for its symmetry.

Essential Reading:

Georges Ifrah, *From One to Zero.*

NCTM, *Historical Topics for the Mathematics Classroom*, Sections II and III.

Supplementary Reading:

Howard Eves, *Great Moments in Mathematics (Before 1650)*, Lecture 15.

Jan Gullberg, *Mathematics—From the Birth of Numbers*, Chapters 3 and 5.

Questions to Consider:

1. Find out what makes a counting number a "perfect" number and what makes a pair of counting numbers "amicable."
2. Create a sequence of four or five numbers. Ask some friends to find the next two numbers in the sequence. Then ask each person to define the pattern of the sequence. You should find that people who give the same answer for the next two numbers may have differing definitions of the pattern.

Lecture Three

Rational Numbers

Scope: This lecture starts with the question, "Can we have something (a symbol) to represent nothing?" Once it is agreed that 0 is a number, the negative integers, which are the opposites of the counting numbers, can be developed. The opposite direction for the negative numbers is established using 0. This leads us to a discussion of the rules for operating with integers and the use of absolute value. The remainder of the lecture is concerned with an understanding of the rational numbers. Ratios can be represented by fractions, decimals, or percents, and we shall concern ourselves with the answers to "why we do what we do" with these numbers. Within our investigation of the rational numbers, we will note some interesting patterns that add to our appreciation of the beauty of mathematics.

Outline

I. For many years, it was felt that there could not be a symbol that represented nothing.

A. In the second century A.D., the Greeks used the letter *omicron* in representing fractions with no parts. *Omicron* is the first letter in the Greek word "*ouden*" which means "nothing."

1. The early Mayans used a zero symbol to act as a placeholder in their base-twenty number system. This was primarily used for recording calendar times and not for computational purposes.
2. The Hindus used a large dot, called *sunya* (meaning empty) as a placeholder. Scholars still are unsure when the *sunya* was first used. Opinions vary from the third century A.D. to the twelfth century.
3. The Arabs adopted the Hindu symbol and called it "*sifr*," which means "vacant." In the thirteenth century, this was Latinized to *zephirum* by Europeans adapting the Hindu-Arabic number system.
4. Over a period of time, the name became "zero" and the large dot grew in size until it became the zero with which we are familiar.
5. In all cases the "vacant" symbol was used as a placeholder. Zero was not used to represent a number by itself because this would be something representing nothing.
6. As the applications of mathematics became more complex, it became necessary to recognize that there was a zero amount. The set of numbers that starts with zero and includes all the counting numbers is called the whole numbers.

B. To understand "negative" numbers, we can use many real-world situations in which numbers can have one of two directions.

1. Examples include earn/spend, profit/loss, up/down, east/west, north/south, and gain/lose.
2. A number on one side of zero would represent the positive direction, such as profit, and the number that is the same distance in the other (or negative) direction from zero would be the opposite of the first number (e.g., loss). Positive two (+2) is the opposite of negative two (-2).
3. The sum of two opposites is zero because they cancel each other. A village that has five babies born and five people die on a particular day has no change in its population (+5 + -5 = 0).
4. Today we have many uses for positive and negative numbers. Some examples are the ups and downs of the Dow-Jones Index®, plus/minus ratings for NHL players, and changes in test scores for schools, school districts, and states.

II. The rules for operating with integers are sometimes taught as if they had been brought down from the mountaintop. Actually the methods used to add, subtract, multiply, and divide integers make a great deal of sense.

A. We will begin with the rules for adding by considering travel on a north/south highway.

1. If I travel 8 miles north and then 10 more miles north, I will be 18 miles north of my starting point.
2. If I travel 12 miles south and then travel an additional 11 miles south, I will be 23 miles south of my starting point.
3. When adding integers with the same sign, keep the sign and add the numbers (+8 + +10 =+18, -12 + -11 = -23).
4. If I travel 12 miles south and then travel 9 miles north, I will be 3 miles south of my starting point.
5. If I travel 13 miles south and then travel 18 miles north, I will be 5 miles north of my starting point.
6. When adding two integers with opposite signs, use the sign of the larger and subtract the smaller from the larger (-12 + + 9 = -3, -13 + +18 = +5).
7. To add more than two integers, combine the positives, then combine the negatives, and finally add the positive sum to the negative sum.

B. To understand subtraction of integers, one must first truly understand the meaning of subtraction.

1. The answer to a subtraction problem, the difference, is really the missing addend, that is, the number that must be added to the number being subtracted to obtain the number from which that number is being subtracted.
2. For example, 8 - 3 = 5 since 3 + 5 = 8

3. Similarly, 8 - 10 = -2 since 10 + -2 = 8.
4. To solve -2 - +6, we must determine what should be added to +6 in order to obtain -2. Clearly the answer is -8 (6 + -8 = -2).
5. To solve 6 - -3, we must determine what should be added to -3 to obtain 6. Clearly the answer is 9 (-3 + 9 = 6).
6. By looking at various examples using the "missing addend" definition of subtraction, you can see that you can subtract an integer by simply adding its opposite. For example, -3 - -9 can be written as -3 + 9, thus the answer to the subtraction problem is 6.

C. Multiplication and division of integers can be easily explained if we investigate simple patterns. The key is to comprehend what is meant by multiplying by a negative number.
1. Clearly 3 x 4 represents 4 + 4 + 4 = 12 and 3 x -4 represents -4 + -4 + -4 = -12.
2. Since the order of numbers being multiplied does not affect the answer (multiplication is commutative), 3 x -4 is identical to -4 x 3. This provides the answer to the meaning of multiplying by a negative number.
3. When multiplying by a negative, multiply and change the sign of the answer. So -8 x -5 is the opposite of 8 x –5, which is the opposite of -40. The answer is 40.
4. The rules for dividing integers are the same as the rules for multiplying. When dividing by a negative, divide and change the sign.

D. When working with signed numbers, there must be a way to indicate the magnitude of the number without regard to sign (direction). The magnitude is indicated by the use of absolute value.
1. The absolute value of -6, written symbolically as |-6|, is 6 since -6 is six units from zero.
2. The absolute value of 6, symbolically |6|, is also 6.
3. Mathematically, the absolute value of an integer is a whole number, but in our common use, the counting numbers and the positive integers are treated as the same numbers.

III. A rational number is one that can be written as the ratio between two integers, provided the second integer is not 0.

A. The rational numbers are comprised of the integers and the fractions.
1. Integers are rational numbers since -4 can be written as -4/1.
2. Rational numbers can be written as fractions, decimals, or percents.

B. The earliest writing of fractions was limited to the reciprocals of counting numbers (1/2, 1/3, etc.).
1. Fractions that were not reciprocals of counting numbers were written as sums of reciprocal fractions. For example, 2/3 was written as the sum of 1/2 and 1/6.

2. The rules for operating with fractions can be a source of confusion unless one realizes that the rules are really the same as the ones used for the counting numbers.
3. The key concept that one must understand is that the denominator of a fraction is its place value.
4. To add or subtract counting numbers, one must align the place values. The same is true for fractions. Only fractions with a common place value, a common denominator, can be added or subtracted.
5. When we multiply counting numbers, we multiply digit times digit and place value times place value. The same is true for multiplying fractions, which is why we multiply numerators and denominators.
6. If working with fractions is a problem for the viewer, I would suggest obtaining *Basic Math*. This Teaching Company series provides a complete investigation of fractions.

C. Once a place value system was used, fractions were written using fractional place values: tenths, hundredths, thousandths, etc.
1. To add or subtract decimals, line up the decimal points, which aligns the place values.
2. Multiplication and division of decimals are the same as the operations with counting numbers. The key to obtaining a correct answer is the location of the decimal point in the answer. Refer to *Basic Math* for a complete investigation of these processes.

D. The simplest form for rational numbers is the percent, because all percents have the same denominator, which is 100.
1. We generally do not concern ourselves with adding or subtracting percents.
2. The percent is a ratio of a number to 100 that is equivalent to the ratio of two other numbers, typically referred to as the "is" and "of" numbers.
3. We have a proportion where the ratio of the percent number to 100 is equal to the ratio of the "is" number to the "of" number. The proportion is solved for the one unknown number.
4. A complete discussion of percents and percent problems can be found in the *Basic Math* series.

IV. There are a number of interesting patterns to be found when one investigates rational numbers.

A. The first pattern involves converting fractions to decimals.
1. Fractions whose denominators have only powers of 2 and 5 as factors can be converted to terminating decimals; e.g., 1/2 = .5, 3/40 = .075, 17/25 = .68, etc.
2. All other fractions have decimal representations that are repeating: 1/3 = .333…, 4/11 = .363636…, 5/6 = .8333…, 5/12 = .41666…, etc.

3. An interesting pattern is found when one examines the decimal representations of the fractions that have a denominator of 7.
4. 1/7 = .142857142857…, 2/7 = .285714285714…, 3/7 = .428571428571…, 4/7 = .571428571428…, 5/7 = .714285714285…, 6/7 = .857142857142…
5. The first pattern that is revealed is that all six repeating decimals have the same six digits in the same order. The only difference is which digit is first.
6. The second pattern is observed if you take the six-digit "cycle" and split it in two equal lengths. For example take 2/7, for which the cycle is 285714. If you split these digits into 285 and 714 and add the two numbers, a sum of 999 is obtained. This is true for all the decimal representations of sevenths.
7. If the fractions that have 13 or 19 in the denominator are written as decimals, the same two patterns are observed. The cycle for each decimal has the same digits in the same order, with the only difference being the first digit. If the cycle for a particular decimal is split in half (there will always be an even number of digits in the cycle), the sum of the two sections will be a string of nines.

B. Another intriguing pattern involves a cycle of five fractions.
1. Select two fractions; we will use 2/5 and 1/3. Take the second number (1/3), add 1 to it and divide that sum by the first number (2/5). The answer to 4/3 divided by 2/5 is 10/3, which is the third number in the sequence.
2. We then take the third number, add 1, and divide that sum by the second number. The result is 13/3 divided by 1/3 is 13, which is the fourth number in the sequence.
3. Add 1 to the fourth number and divide that sum by the third number. In our example, the answer is 21/5 (14 divided by 10/3).
4. If 1 is added to the fifth number in the sequence (21/5) and that sum is divided by the fourth number, the answer is 2/5, which happens to be the first number (26/5 divided by 13).
5. No matter which fractions are chosen for the first two numbers, the five-number cycle ALWAYS leads back to the first fraction.
6. Later in the course (Lecture Six), we shall prove algebraically why this is so.

C. There is an interesting method to convert a repeating decimal to a fraction.
1. Because all rational numbers can be written as a terminating decimal or a repeating decimal, given a repeating decimal, we should be able to find its fractional equivalent.
2. First, we will examine a decimal that begins its cycle immediately. An example is .376376… Remember that the 376 cycle goes on forever, that is, it has infinite length.

3. If we multiply the decimal by 1000, we get 376.376376… If n is the fraction we seek , then we know that 1000n equals 376.376376…
4. By subtracting the original decimal from 1000n, we obtain 999n = 376. That is, 1000n minus n equals 999n and 376.376376… minus .376376… equals 376.
5. Thus, n equals 376/999. This fraction cannot be simplified and is the fractional equivalent to our repeating decimal.
6. If the repeating decimal does not begin its cycle immediately, there is a slight modification to the method. We will use .45252… as our example.
7. We multiply the decimal by 1000 to get the first cycle in front of the decimal point, which gives us 1000n = 452.5252…
8. We then multiply the original decimal by 10 to move the non-repeating part of the decimal to the left of the decimal point, which gives us 10n = 4.5252…
9. Subtracting 1000n - 10n gives us 990n = 448. Thus, n = 448/990, which can be simplified, giving us the fractional equivalent of .45252… as 224/495.

Essential Reading:

Georges Ifrah, *From One to Zero*.

NCTM, *Historical Topics for the Mathematics Classroom*, Section II.

Supplementary Reading:

Jan Gullberg, *Mathematics—From the Birth of Numbers*, Chapters 2 and 3.

Questions to Consider:

1. What type of number would have a decimal equivalent that neither terminated nor had a fixed repeating cycle (an example would be .101001000100001…)?
2. What is the actual value of the repeating decimal .999999? Most people will not believe that the answer is 1. First prove to yourself that this is true. Then test the question and your proof on friends who are not mathematically inclined.

Lecture Four

Numbers That Are Not Rational

Scope: In our first three lectures, we investigated the development of numbers that can be measured: counting numbers, zero, integers, and the rational numbers. Now we progress to numbers that are more abstract. The irrational numbers are numbers that cannot be written as the ratio of two integers. Our first task will be to review the struggle of the ancients to believe that there were numbers that could not be measured. These numbers were called incommensurable. We will look at the development of the symbology for the square root. Then we will examine the development of the number pi for the ratio between the circumference and diameter of a circle and the natural base "e," which will include the concept of limit. The last part of this lecture will involve a deliberate look at the arithmetic of square roots.

Outline

I. In the sixth century B.C., the Pythagoreans believed that ALL existing things could be represented by a whole number. In an ironic way, the theorem named for the Pythagoreans led to the proof that this belief was untrue.

   A. The Pythagorean theorem states that the sum of the squares of the two legs of a right triangle is equal to the square of the hypotenuse. Aristotle provided the first known proof of the existence of "incommensurable" numbers (numbers without comparison).

      1. Aristotle proposed a square whose side had length n (n being a counting number) and whose diagonal had length m (m also being a counting number). The ratio of m to n, represented by the fraction m/n, could not be reduced—that is, m and n had no common divisor other than 1.
      2. Using the Pythagorean theorem, Aristotle showed that $m^2 = n^2 + n^2 = 2n^2$.
      3. Next he showed that m^2 must be an even number because it is equal to 2 times the square of n and n is a counting number.
      4. Since m and n share no common divisor other than 1, if m is even, then n must be odd.
      5. Because m is an even number, we can define m = 2h, where h is a counting number.
      6. Now m^2 must equal the square of 2h, which is $4h^2$. Since $m^2 = 2n^2$ and $m^2 = 4h^2$, $2n^2 = 4h^2$. This means that $n^2 = 2h^2$, thus n^2 is an even number.

      7. If n^2 is even, then n must be an even number. This is a contradiction of the original statement that m and n have no common divisor other than 1.
      8. This contradiction means that the original supposition that m/n was a rational number (the ratio of two integers) is not true. Thus, Aristotle used the Pythagorean theorem to disprove the Pythagorean belief that all numbers were commensurable.

   B. Once we understand that irrational numbers exist, we can begin to look at the various types of irrational numbers. A common type of irrational number is the square root of a number that is not square.

      1. It is interesting to note that from 1 through 10,000, there are only one hundred square numbers. From 10,001 through 40,000, there are only one hundred square numbers.
      2. The number of counting numbers that are not square far exceeds the number of square counting numbers. Thus, there a large number of irrationals that are equal to the square roots of these "unsquare" counting numbers.
      3. One interesting pattern that exists when investigating square numbers concerns which digits can be in the ones place of a square number. It is impossible to multiply a digit by itself and obtain a number that ends in 2, 3, 7, or 8.
      4. A look at this pattern leads us to another proof of the existence of irrational numbers.
      5. The square root of 2 must exceed 1 and be less than 2. If it is rational and is not a whole number, then it must be a decimal. The last place of the decimal multiplied by itself must be 0, because when we square the square root of 2, we must get 2.000000…
      6. Zero is the only digit whose square ends in 0, and the answer cannot be 1.000000…

   C. Other irrationals are roots beyond the square root.

      1. While it takes 10,000 counting numbers to produce the first one hundred square counting numbers, it takes 1,000,000 counting numbers to produce the first one hundred cubic counting numbers.
      2. Thus, in the first 1,000,000 counting numbers, there are 999,900 that do not have rational cube roots.
      3. There are even more irrationals created by taking the fourth root of numbers that are not equal to the fourth power of a counting number.
      4. And fifth roots, and sixth roots, and so on give us an infinite set of irrational numbers.

   D. As humans investigated the application of mathematics, important irrational numbers were discovered.

      1. The universal ratio between the circumference and diameter of a circle is irrational.

2. The natural base for many exponential functions is irrational.

II. The use of irrational numbers led to the development of some very common mathematical symbols. Before the sixteenth century, mathematics problems were written using words. Slowly, the use of abbreviations was accepted. This led to the introduction of shorthand symbols.

A. One of the first symbols developed in Europe was the radical symbol for square roots.

1. The Latin word for root is *radix*, and from *radix* the square root symbol was formed over a period of time.
2. The first abbreviation for *radix* was the R-combined-with-x symbol that we recognize as the prescription symbol today.
3. Initially a 2 was included in the symbol to indicate square root.
4. Since the square root was the most common type of root, the 2 was dropped from the symbol and, in 1525, the Rx was replaced with a symbol resembling a check mark (•).
5. In 1553, the symbol was changed to what looks like a small z with an extension.
6. By the 1600s, the radical symbol that we use today was in common use. It was based on a fancy version of the letter "r" for *radix*.

B. The ratio between the circumference and the diameter of a circle had been sought for centuries, but in the eighteenth century, a universal symbol was finally adopted.

1. The Babylonians thought the ratio was equal to 3 1/8. That was in 1700 B.C.
2. In I Kings 7:23, while describing King Solomon's temple (1000 B.C.), the ratio is indicated as equal to 3. There has been some research to indicate that the use of two different Hebrew spellings of a word in the description might indicate a more accurate estimate of the ratio.
3. In the third century B.C., Archimedes believed the ratio to be between 3 1/70 and 3 1/7.
4. In China in the year 470 A.D., the ratio was written as 355/113. This translates as a decimal to 3.1415929…
5. Forty years later, in India, the ratio was written as 62,832/20,000, which is 3.1416.
6. In 1736, Leonhard Euler used the Greek letter "pi" (Π) to symbolize the ratio and we have used this ever since.
7. Pi is an irrational number because its decimal representation neither terminates nor has a repeating cycle.
8. The digits in pi have no pattern and have been used to create random numbers.
9. In an early science fiction movie, pi is used to communicate with Martians. The use of pi was suggested because any scientific culture would be familiar with the ratio.

C. As the application of mathematics became more complex, exponential forms of numbers became very useful. The discovery of a base that was found in various settings required a symbol.

1. This natural base was first recognized in 1618 by John Napier. Napier is given credit for the development of tables of logarithms.
2. In 1728, Euler represented the natural base as "e" and defined it as the infinite sum of 1 + 1/1 + 1/1x2 + 1/1x2x3 + 1/1x2x3x4 +…
3. Euler also defined e as the limit, as n goes to infinity, of $(1 + 1/n)^n$.
4. The decimal representation of e is 2.718281… It does not terminate and does not have a repeating cycle.
5. Euler combined the most significant symbols in mathematics in one equation: $e^{i\pi} + 1 = 0$. The "i" is used for the square root of -1 and will be discussed at length in the next lecture.

III. The use of common square roots required the development of rules for operating with these numbers. To many, these operating rules are different than the rules for counting numbers but actually they are based on the same definition of addition and multiplication.

A. You can only add or subtract numbers if they have the same place value. Look at the square root as a place value.

1. The square root of 2 and the square root of 8 cannot be added or subtracted since they are different place values.
2. Square roots can be simplified by seeking a square number that is a factor of the number inside the square root. The square root of 8 can be broken down into the square root of 4 times the square root of 2. This is equal to 2 times the square root of 2.
3. Because of simplification, the square root of 8 and the square root of 2 can be added. The square root of 8 is 2 times the square root of 2. Adding the square root of 2 and 2 times the square root of 2 gives an answer of 3 times the square root of 2.
4. Subtraction is accomplished in the same way. The square root of 50 minus the square root of 18 is equal to 2 times the square root of 2 once the original square roots are simplified.

B. Multiplication, as before, requires the multiplication of the digits and the place values.

1. The square root of 3 times the square root of 7 equals the square root of 21.
2. Three times the square root of 6 multiplied by 2 times the square root of 2 has a product of 6 times the square root of 12.
3. Since the square root of 12 is 2 times the square root of 3, the answer can be simplified to 12 times the square root of 3.
4. Note that multiplying a square root by itself produces a rational number. Multiplying a two-term irrational number (called a "surd")

by its conjugate also produces a rational number (e.g., • 5 + • 2 and its conjugate • 2 + • 5).

5. A division problem involving square roots is solved by writing the problem as a fraction. The denominator is rationalized and the numerator is multiplied by the same number that was used to rationalize the denominator. The fraction formed is a rational number.

Essential Reading:

NCTM, *Historical Topics for the Mathematics Classroom*, Section II.

Jan Gullberg, *Mathematics—From the Birth of Numbers*, Chapters 2 and 3.

Supplementary Reading:

Howard Eves, *Great Moments in Mathematics (Before 1650).*

Sanderson Smith, *Agnesi to Zeno*.

Questions to Consider:

1. In looking for numbers that are perfect cubes, what digits appearing in the ones place of a counting number would indicate that the number could not be a cubic number?
2. How would you respond to a person who says that the square root of 2 can be written as a decimal number and demonstrates that belief by showing you the decimal representation of the square root of 2 on a calculator?

Lecture Five

Imaginary and Complex Numbers

Scope: As humans progressed into more abstract uses for numbers, there arose a need to recognize numbers that when multiplied by themselves, produced negative squares. These numbers became the imaginary numbers, and "i" was used for the square root of -1. When real and imaginary numbers were added, complex numbers were produced. This lecture covers the history of the development of complex numbers and the arithmetic of these numbers. The geometric representation of complex numbers led to the discovery of complex roots of real numbers.

Outline

I. The imaginary numbers are much newer than the irrationals, because the belief that such numbers should be created required the use of mathematics in more abstract situations. Once imaginary numbers were used, the creation of the complex numbers was inevitable.

   A. The first known writing that hinted at the existence of the imaginary numbers was in 50 A.D.

      1. Heron of Alexandria wrote a number that today would be called the square root of the difference 81 - 144.
      2. Since 81 - 144 is -63, Heron was writing about the square root of a negative number.
      3. Numbers that were the square roots of negatives were eventually called "subtle" numbers.
      4. In 1637, René Descartes first used the terms "real number" and "imaginary number" to differentiate between numbers that if squared produced 0 or positive numbers and numbers that when squared produce negative numbers.
      5. Euler, who we have seen was responsible for a number of mathematical symbols, defined "i" as the square root of -1 in 1777.

   B. Once imaginary numbers came into common use, it was a natural consequence for a combination of a real number and an imaginary number to be used in mathematics.

      1. Geralamo Cardano posed a problem in 1545. He asked for two numbers whose sum is 10 and whose product is 40. The answer is 5 + the square root of -15 and 5 - the square root of -15.
      2. Finally, in 1831, Carl Gauss used the term "complex number" and wrote the form of that number as a + bi, where a and b are real numbers and i is the square root of -1.

3. A real number would be a + 0i, and a purely imaginary number would be 0 + bi.
4. It was discovered that multiplying (a + bi) x (a - bi) had a product of $a^2 + b^2$, which is a real number.
5. In 1821, Augustin Louis Cauchy was the first to use the term "conjugate" for a + bi and a - bi.

II. The development of complex numbers required algorithms for the addition, subtraction, multiplication, and division of such numbers.

A. The rules of addition require that one can only add numbers with the same place value.
1. When adding complex numbers, add the real parts of each number to get the real part of the sum. Add the imaginary parts of each number to get the imaginary part of the sum.
2. (8 + 3i) + (6 - 7i) = 14 - 4i.
3. (9 + 8i) + (-11 + 3i) + (4 - 7i) = 2 + 4i.

B. The rules for subtraction are always the same as the rules for addition.
1. (9 + 6i) - (7 + 2i) = 2 + 4i.
2. (-7 + 12i) - (6 - 3i) = -13 + 15i.

C. In multiplying complex numbers, each part of one number is multiplied by each part of the other number.
1. The FOIL method from your high school algebra class works here.
2. $(2 + 3i)(4 - 7i) = 8 - 14i + 12i - 21i^2 = 8 - 14i + 12i + 21$ (since i^2 is -1).
3. Combining like terms, we get an answer of 29 - 2i.
4. Note that a positive imaginary multiplied by a negative imaginary has a positive real product.
5. Squaring a complex number is accomplished the same way: $(2 + i)^2 = 4 + 2i + 2i - 1 = 3 + 4i$.
6. In this case, we see that a positive imaginary multiplied by a positive imaginary has a negative real product.
7. Multiplying a complex number by its conjugate yields a product that is real: (3 + 6i)(3 - 6i) = 9 + 36 = 45.

D. Division of complex numbers is accomplished by writing the division problem as a fraction and using the conjugate to obtain a real denominator.
1. (2 + 3i) ÷ (1 + i) is written as (2 + 3i)/(1 + i).
2. We multiply the numerator and the denominator of the fraction by the conjugate of the denominator, which is (1 -i).
3. The new fraction is (2 + 3i - 2i + 3)/(1 + 1) = (5 + i)/2 or 2.5 + .5i.

III. The use of complex numbers led to new answers to old problems.

A. If there are two square roots of a number, why are there not three cube roots for a number?

1. In 1629, Albert Girard declared that there should be three answers for the cube root of 1.
2. In 1730, Abraham de Moivre produced a method to find the two other cube roots of 1 other than 1.
3. His method led to the conjugate answers of .5 + .5i times the square root of 3 and .5 - .5i times the square root of 3.
4. Squaring .5 + .5i times the square root of 3 gives a product of -.5 - .5i times the square root of 3. Multiplying this by .5 + .5i times the square root of 3 gives the cube, which is .25 + .75, which is 1.
5. The conjugate when cubed also is equal to 1.

B. A graphical representation of complex numbers is helpful in finding patterns.
1. The complex number a + bi is written as an ordered pair (a, b), where the first coordinate of the ordered pair is the real part of the complex number and the second coordinate is the real coefficient of the imaginary term.
2. The point (a, b) is plotted and that point is a graphical representation of the complex number.
3. It is immediately apparent that conjugates are reflections about the x-axis.
4. If we look at the graphical representation of the three cube roots of 1, we see that the three points have equal angular separation and that the distance from the origin to each point is the same.
5. The angular separation is 120 degrees (360 divided by 3), and the distance from the origin to each point is 1.
6. The three points are (1, 0), (-.5, .5i times the square root of 3), and (-.5, -.5i times the square root of 3).
7. The distance from the origin to the point is the magnitude of the complex number and is the square root of the sum of the squares of a and b (using the Pythagorean theorem).

C. Using the graphical representation of complex numbers we can find the four fourth roots of 1.
1. The four answers should have a magnitude of 1 and should have an angular separation of one fourth of 360 degrees.
2. Since we know that 1 and -1 are fourth roots of 1 and these two real numbers would have graphical representations of (1, 0) and (-1, 0), the other two points should be located on the y-axis.
3. The missing points would be at (0, 1) and (0, -1); thus, the other fourth roots of 1 are i and -i. Indeed the fourth power of i and of -i are both equal to 1.

IV. Complex numbers are used for practical purposes when analyzing three-phase alternating current. A discussion of that will be left for your personal investigation or a future lecture series on electrical engineering.

Essential Reading:

NCTM, *Historical Topics for the Mathematics Classroom*, Section V.

Jan Gullberg, *Mathematics—From the Birth of Numbers*, Chapter 3.

Supplementary Reading:

Lynn Arthur Steen, *On the Shoulders of Giants*, QUANTITY.

Questions to Consider:

1. We have seen that Leonhard Euler was responsible for defining the symbols for many important numbers. Find out more about this prolific mathematician and decide why his symbols were universally accepted.
2. Speak with an electrical engineer and have him or her explain how complex numbers are used to analyze three-phase alternating current.

Lecture Six

Algebra: Generalizing Arithmetic

Scope: To many students, algebra is a series of symbols and procedures that are unrelated to anything the student has seen before. In this lecture we will see how algebra is simply a generalization of arithmetic. By understanding arithmetic, we are better able to understand algebra. Our knowledge of algebra helps elevate our understanding of the concepts behind arithmetic. This lecture will discuss the development of algebraic symbology, specifically related to the polynomial. The polynomial will be the vehicle to demonstrate the relationship between algebra and arithmetic. The methods of adding, subtracting, multiplying, and dividing polynomials will be investigated, including the rationale for "synthetic division." Finally, algebra will be used to demystify two arithmetic tricks.

Outline

I. The key to understanding algebra is one's conceptual understanding of the variable and its use in the polynomial.

A. A variable is simply a symbol that stands for a missing or unknown number.

1. A perusal of elementary school mathematics texts will show that variables are introduced early in the education process. Problems such as 5 + blue square = 8 or 10 - red triangles = 7 use brightly colored geometric shapes for variables.
2. Typically, letters of the alphabet are used as variables with the last three letters (x, y, and z) used most often for unknown quantities.
3. Actually any symbol other that a digit or an operations symbol (+, -, etc.) may be used as a variable. Letters are typically used because they are easy to write and are commonly recognized.
4. In early mathematical writings, words or abbreviations were used to express mathematical relationships.
5. In the fourth century A.D., there was early use of symbology in Greek writing. K^{υ} was used to represent the Greek word K• • • • , which meant the cube or third power of a number.
6. The fourth century Greek mathematicians used /|\ Δ^{υ} to symbolize the negative of the square of a number.
7. The more general use of polynomials in the sixteenth century led to the adoption of the notation we use today.

B. A polynomial is the sum of one or more terms with each term having a real number multiplied by a variable that is raised to a whole number power.

1. Each term of a polynomial is called a monomial. A monomial can be a number such as 3, .5, or -2.1. It can be a whole number power of a variable, such as x, -y or z^2. It can also be a term such as 5x or $-3z^2$.
2. The degree of a polynomial is the highest power of a variable in the polynomial. The degree of the polynomial $x^3 - 14x^2 + 7x - 100$ is three.
3. The lead coefficient is the coefficient of the monomial that determines the degree.
4. We will focus on polynomials that use only one variable. That variable will be x.

C. In the sixteenth century, scholars struggled to adapt a universal system for writing variable expressions.

1. In 1557, "=" was first used to symbolize "equals."
2. In 1591, Francois Viete wrote a polynomial equation as IQC - 15QQ + 85C - 225Q + 274N *aequatur* 120.
3. That equation would be written today as $x^6 - 15x^4 + 85x^3 - 225x^2 + 274x = 120$.
4. By 1637, René Descartes was writing an equation such as $x^3 - 6xx + 13x - 10 = 0$.
5. Isaac Newton established that the second power of x should be written as x^2 rather than as xx.

II. It is best if polynomials are seen as whole numbers whose place values are powers of x rather than powers of 10.

A. Looking at a polynomial as a generalized whole number helps us understand place value.

1. Just as we can write the whole number 237, we can write the polynomial $2x^2 + 3x + 7$. When x is 10, then $2x^2 + 3x + 7$ is equal to 237.
2. Because the place value expressed as a power of x is shown rather than understood by the position of a digit, the coefficient, which is a generalization of the digit of a whole number, can be any type of real number.
3. Thus, we can write $2x^2 - 3x + 7$. Consider the number 2 -37—two hundred, negative thirty, seven. What amount does this represent?
4. Since 2 -37 shows 2 hundreds, -3 tens, and 7 ones, it must be equal to 177.
5. We can write the polynomial $2x^2 + 1/2\ x + 7$. Consider the number 2 1/2 7. I call this number two hundred "halfty" seven. What amount does this represent?
6. This number contains 2 hundreds, half a ten, and 7 ones, so it must be equal to 212 (200 + 5 + 7).

B. There is no need for zeros as placeholders in the writing of a polynomial.

1. We can write $2x^3 + 25x + 12$ without showing the second power of x.
2. If desired, we can write the same polynomial as $2x^3 + 0x^2 + 25x + 12$.

III. The methods for adding, subtracting, multiplying, and dividing polynomials make a great deal of sense when one relates the polynomial to the whole number.

A. To add two or more polynomials, align the place values by lining up the powers of x.

1. Add the coefficients for each power of x, placing the answer in the same place value.
2. Adding $(2x^2 + 3x + 7) + (3x^2 - 9x - 1)$ we obtain $5x^2 - 6x + 6$.
3. Replacing x with 10 can be used to check the answer: $2x^2 + 3x + 7$ is 237, while $3x^2 - 9x - 1$ is 209.
4. Our answer, $5x^2 - 6x + 6$, equals 446, which is the sum of 237 and 209.

B. To subtract polynomials, change all the signs in the second polynomial and add the polynomials.

1. $(2x^2 + 3x + 7) - (x^2 + 7x - 8) = (2x^2 + 3x + 7) + (-x^2 - 7x + 8) = x^2 - 4x + 15$.
2. Subtraction problems can be checked by substituting 10 for x. Since 237 - 162 = 75 and $x^2 - 4x + 15$ equals 75, we know that our answer is correct.

C. Multiplication of polynomials is accomplished using the same techniques used to multiply multi-digit whole numbers.

1. Each monomial in one polynomial is multiplied by each monomial in the other polynomial and the products are added along place values to obtain the final answer.
2. As with adding and subtracting, we can check our answer by substituting 10 for x.

D. Dividing polynomials uses the same steps as the long division algorithm.

1. We will restrict ourselves to dividing using binomials of the form x – a, where a is a positive or negative integer.
2. We start with the highest power term in the dividend and divide it by x.
3. We then multiply the answer to that division problem by the divisor. This product is subtracted from the original dividend by changing the signs of the second polynomial and adding.
4. This process is repeated until a remainder is obtained.
5. As with the other types of problems, we can check our answer by substituting 10 for x.

6. The synthetic division method is based on the division algorithm. To divide $x^2 - 3x + 7$ by $x + 2$, we write the problem this way:

$$\begin{array}{r|rrr} \underline{-2} & 1 & -3 & 7 \\ & & -2 & 10 \\ \hline & 1 & -5 & \underline{17} \end{array}$$

which is $x - 5 + 17/(x+2)$.

IV. Algebra can be used to investigate why arithmetic "tricks" work all the time.

A. There is a "trick" called "Birthday Magic."

1. Start with your birth month (1 for January, 2 for February, etc.) and multiply that number by 4. Then add 16 and multiply that sum by 5. Subtract 7 and multiply the result by 5. Add the day of the month of your birthday and subtract 365. The result should be your birthday.
2. For example if your birthday was October 23, you would start with 10 and multiply by 4, which is 40. Forty plus 16 is 56, which is then multiplied by 5, giving a product of 280. This number minus 7 is 273, which is then multiplied by 5 to get 1365. When the day of the month of the birthday (23) is added to 1365 the sum is 1388. When 365 is subtracted, the final result is 1023.
3. Is this magic? Algebra demonstrates why it works. If we call our answer A, the algebraic formula for obtaining A is $A = 5 [5 (4m + 16) - 7] + d - 365$, where m is the month and d is the day.
4. If we simplify the process: $5 [5 (4m + 16) - 7] + d - 365 = 100m + 365 + d - 365$, which is $100m + d$.
5. If $m = 1$ and $d = 15$ (January 15), $100m + d$ is 115. If $m = 11$ and $d = 18$ (November 18), $100m + d = 1118$.

B. In Lecture Three, we saw a pattern that involved a cycle of five fractions.

1. We select two fractions. The third number in the sequence is one more than the second number divided by the first number. We continue this process and the number after the fifth number is the first number, so we have a five-number cycle.
2. If we call the first number a and the second number b, the third number is $(b + 1)/a$. The fourth number, which is $(b + 1) /a + 1$ all divided by b, simplifies into $(a + b + 1)/(ab)$.
3. The fifth number is $(ab + a + b + 1)/(b^2 + b + 1)$. When 1 is added and the result is divided by $(a + b + 1)/(ab)$, the result is a complex-looking fraction that simplifies to be a, the first number in the cycle.
4. Algebra demonstrates that no matter what numbers you choose for a and b, the five-number cycle will always work.

Essential Reading:

NCTM, *Historical Topics for the Mathematics Classroom*, Section V.

Jan Gullberg, *Mathematics—From the Birth of Numbers*, Chapter 4.

Supplementary Reading:

Lynn Arthur Steen, *On the Shoulders of Giants*, QUANTITY.

Questions to Consider:

1. Review the process of factoring polynomials (you may want to view the appropriate lessons in The Teaching Company's *Algebra I* or *Algebra II* series) and decide what is the arithmetic analog to factoring, and why do we need to learn how to factor?
2. Look at Viete's method of writing a polynomial and determine the derivation of each of his symbols.

Lecture Seven

The Linear Function

Scope: Much of the time spent in high school algebra classes (and even a considerable amount of time in many college mathematics classes) is spent writing, solving, and graphing linear functions. These are equations that can be written as $y = mx + b$ or $Ax + By = C$. Much of the study of linear functions is done in an abstract fashion, leaving students to grope for the meaning and importance of what they have studied. This lecture will attempt to clarify both the importance and the meaning of the linear function. First, we will obtain ordered pairs of (x, y) that will satisfy a certain equation. We will examine patterns that allow us to discover the meaning of the slope and the intercepts. Then, by graphing the equation, we will have a geometric image of the equation that provides a visual picture of the meaning of the slope and intercepts. Finally, we will examine the use of linear functions as mathematical models for real-world situations, demonstrating the vital importance of linear functions in the lives of all citizens.

Outline

I. In 1637, Rene Descartes published *Geometrie*, in which he created a method for producing a geometric picture of an algebraic equation. In effect, Descartes had invented graph paper by constructing two number lines that intersect at right angles at zero.

A. Graph paper represents what is called the Cartesian plane, the eponymous adjective is used to honor Descartes, although a number of other seventeenth-century scholars, mostly French, contributed to the development of the graphing of equations.

1. We will begin with an equation, $y = 2x + 4$. By substituting various values for x, we obtain the output, which is y. For example if x is 9, $2x + 4$ is $18 + 4 = 22$.
2. We can make a roster of these ordered pairs (x, y) so that each ordered pair satisfies the equation—substituting the values for x and y provides a "true" equation.
3. Using -3, -2, -1, 0, 1, 2 and 3 for x, we obtain the following roster of ordered pairs:

x	y
-3	-2
-2	0
-1	2
0	4
1	6
2	8
3	10

4. There are two important ordered pairs that can be noted. When x is -2, y is 0 and when x is 0, y is 4.
5. An extremely important pattern can be observed if we move down the roster. Every time x increases by 1, y increases by 2. It happens for every value of x. If we increase x by 2, say from -2 to 0, y increases by 4 (twice 2), and if we increase x by 5, y increases by 10 (5 times 2).
6. If we try another equation, such as $y = 5x - 10$, we notice that when x is 2, y is 0; when x is 0, y is –10; and if we increase x by 1, y increases by 5.
7. If we examine a third equation, $y = -3x -6$, we see that when x is -2, y is 0; when x is 0, y is –6; and if we increase x by 1, y decreases by 3.

B. The value of x that causes y to be 0 is called the x-intercept, for which there is no special mathematical symbol.

1. If we write the general equation for a linear function (Descartes was the first to use the term "function" to describe an equation) as $y = mx + b$ and solve for x if y is 0, we obtain the x-intercept, which is equal to $-b/m$.
2. In $y = 2x + 4$, the x-intercept should be at $x = -2$ and it is.
3. In $y = 5x – 10$, the x-intercept should be at $x = 2$ and it is.
4. In $y = -3x –6$, the x-intercept should be at $x = -2$ and it is.

C. The value of y when x is 0 is called the y-intercept. The lower case letter "b" is the universal symbol for the y-intercept.

1. If we write the general equation for the linear function as $y = mx + b$, then when x is 0 y will equal b.
2. In $y = 2x + 4$, the y-intercept should be at $x = 4$ and it is.
3. In $y = 5x – 10$, the y-intercept should be at $x = -10$ and it is.
4. In $y = -3x –6$, the y-intercept should be at $x = -6$ and it is.

D. The amount y changes when x increases by 1 is called the slope. The symbol for slope is "m."

1. There are varying stories about the use of m for slope. Most probably it was the first letter in a Latin or French word relating to change (perhaps the Latin *mutare*—to change). As noted

previously, many of the mathematicians who worked on the graphical display of algebraic equations in the seventeenth century were French and they generally wrote their scholarly discourses in Latin.

2. Examination of the three linear equations previously analyzed reveals that indeed the number in front of the x was the amount that y changed when x increased by 1.
3. In general, the slope can be found using the formula m = the change in y divided by the change in x.

II. We can now return to our roster of ordered pairs for y = 2x + 4 and graph the points in the Cartesian plane that correspond to those ordered pairs.

A. Each ordered point has a unique location on the graph paper. A point is marked at the location for each of the points in our roster.

1. We note that all the points seem to align themselves in a straight line.
2. Since the slope indicates how much y changes if x increases by 1, the change in y from point to point is a constant, giving us the apparent straight line.
3. We can now draw a straight line through the points and extend that line at the same slope in both directions.
4. Every point on that line has (x, y) coordinates that will work in the equation. The points may have x and/or y values that are fractions or irrationals, but those values will work in the equation.
5. The coordinates of any point NOT on the line will not work in the equation.
6. Because the graph of any equation of the form y = mx + b is a straight line, these equations are classified as "linear."

B. There are two special types of linear equations that involve the use of only one variable.

1. y = k, where k is a constant, is a horizontal line, because each point must have the same y.
2. The slope of a horizontal line is 0, because when x increases by 1, y remains constant.
3. x = k, where k is a constant, is a vertical line, because each point must have the same x.
4. The slope of a vertical line is undefined, or is said not to exist, because x cannot change.

III. Using linear functions as models for real-world phenomena demonstrates the vital importance of these equations.

A. We will first investigate a postage model. Let us assume that to mail a letter, you must pay 50 cents for the first ounce and 25 cents for each additional ounce (or fraction).

1. The y-intercept, b, is the value of x when y is 0. If our item to be mailed weighs 1 ounce or less, the cost is 50 cents. If we define x as the number of ounces beyond the first ounce, then it is clear that the y-intercept is 50.
2. Every time we add an ounce to our mailing, the cost increases by 25 cents; thus, the slope must be 25.
3. We now have a linear model y = 25x + 50, which will give us the postage required (y) for weight in ounces, beyond the first ounce (x).
4. The model can be used to find the postage for a 10-ounce package. Since 10 is 9 more than1, x = 9. The value of y when x is 9 is 275. Thus the 10-ounce mailing will require $2.75 in postage.
5. The model can also be used to find out how much something weighed if the postage required to mail it is $4.00. Replacing y with 400, we have 400 = 25x + 50. Thus x is 14, which means the package weighed 15 ounces.

B. We will next investigate a revenue and expense model.

1. If x is the number of items sold and each is sold for $100, the revenue model is y = 100x. The slope is 100 because selling one more item increases revenues by $100. The y-intercept is 0 because if no items are sold, there is no revenue.
2. If the direct cost to produce and sell each item is $50 and the fixed cost (or overhead) is $1000, then the expense model is y = 50x + 1000.
3. If we graph the two equations, we find that they intersect at x = 20, y = 2000.
4. This intersection point is quite significant—it is called the "breakeven" point. If we sell exactly twenty items we neither make a profit nor take a loss.
5. If we sell fewer than twenty items, we are losing money and if we sell more than twenty, items we are making a profit.
6. The amount of profit or loss (negative value for y) can be determined from the model if the x is known.
7. The number of items that must be sold to generate a specific amount of profit can be determined from the model by substituting the desired amount of profit for y and solving for x.
8. It should be noted that we can solve for the breakeven point algebraically by setting the right side of the revenue model equal to the right side of the expense model: 100x = 50x + 1000. The solution is x = 20.

C. We can use linear models to compare various cellular telephone plans.

1. Let us assume that there are four plans available: $20 per month with unlimited free minutes, $2 per minute with no monthly charge, $1.50 per minute with a $5 monthly charge, and $1 per minute with a $10 monthly charge.

2. Each plan can be written as a linear equation. The first is $y = 20$, the second is $y = 2x$, the third is $y = 1.50x + 5$, and the last is $y = x + 10$.
3. If we graph the four equations, it is apparent that if the consumer uses exactly 10 minutes per month, all the plans have the same cost.
4. From the graph, we can easily see which plan is best for which amount of monthly minutes.

IV. In the real world, many models that are used are approximations because all the points do not really line up to form a line.

A. If we examine data on fat content and calories in pizza slices, we can form a model to predict calories (y) based on fat content in grams (x).
1. One data point is (4, 275) and another is (19, 383).
2. If we divide the change in y (383 - 275) by the change in x (19 - 4), we obtain a slope of 7.2.
3. From the graphical display of the data, it appears that a reasonable estimate for the y-intercept is 250. Thus, our model is $y = 7.2x + 250$. We can refine the equation on the graphing calculator to improve the fit of the model.
4. The slope of 7.2 means that if the grams of fat in a slice of pizza increase by 1, we would expect an increase of 7.2 calories.
5. The y-intercept of 250 means that if there was no fat in the pizza, we would still expect 250 calories. These calories would be derived from the carbohydrates and protein in the pizza.

B. If we examine data relating national cigarette consumption to coronary heart disease (CHD) mortality rates, we can form a linear model to quantify that relationship.
1. The x values are the average number of cigarettes smoked per year per adult by country, while the y values are the number of adult deaths from CHD per 100,000 in the adult population (ages 35–64).
2. We select two points at (3900, 257) and at (2790, 194). The first point represents the United States.
3. The slope is computed to be .057, which means if every adult smokes one more cigarette per year, the number of deaths from CHD will increase by .057 per 100,000.
4. From the graph, we estimate the y-intercept to be 25, which means that if there were no cigarettes smoked in a country, there would still be 25 deaths due to CHD per 100,000 adults.
5. The model would be written as $y = .057x + 25$.

Essential Reading:

William Dunham, *The Mathematical Universe.*

Roland Larson, Timothy Kanold, and Lee Stiff, *Algebra 1—AN INTEGRATED APPROACH.*

Supplementary Reading:

E. T. Bell, *Men of Mathematics.*

Gail Burrill et al., *Data Analysis and Statistics across the Curriculum.*

NCTM, *Historical Topics for the Mathematics Classroom*, Section V.

Questions to Consider:

1. Gather some two-variable data, plot the points, and determine a reasonable linear model. Define the meaning of the slope and the y-intercept.
2. Think back to your personal experience in high school mathematics. Express the difference between how material on linear equations was presented in those classes and how it has been presented in Lecture Seven.

Lecture Eight

Quadratic and Cubic Functions

Scope: Not all models are best represented by a linear function. A very common type of model is a second-degree polynomial or quadratic function. Another useful type of function is the third-degree polynomial or cubic function. This lecture will investigate the solution to the general quadratic equation, as well as the technique for graphing quadratic functions. Real-world situations that are best modeled using a quadratic function will be discussed. Two different forms of the quadratic function will be explored. Applications of the cubic function will be examined. The importance of a special point on a typical cubic graph will be analyzed with regard to its application in timeline data. The history of the search for a general solution to the cubic equation will be related, including the very interesting and tragic life of the brilliant mathematician responsible for discovering the general solution.

Outline

I. If the world was simply linear, we would have no need for calculus. Although linear equations are exceptionally helpful, there are numerous situations that require a model in which the change in y is not constant (i.e., the slope is not constant or linear). One very common non-linear model is the second-degree polynomial or quadratic function.

- **A.** There are many physical properties that vary with the second power (or square) of some measurement.
  1. The force of gravity is proportional to the inverse of the square of the distance between the two bodies.
  2. The intensity of light is a function of the inverse of the square of the distance between the light source and the body being illuminated.
  3. The arc of a circle is proportional to the square of the radius.
  4. The distance a body travels in free fall under gravity is equal to 1/2 gt^2, where g is the acceleration due to gravity and t is the time the body has been in free fall.
  5. The lift of an airplane wing is proportional to the square of the airplane's speed. You might notice that faster airplanes generally have smaller wings.
- **B.** For hundreds of years, a general solution to the quadratic equation $ax^2 + bx + c = 0$ was sought.
  1. Early investigation of the general solution was done by Brahmagupta in India in 628.

  2. In 825, the Arab mathematician al-Khowarizmi worked on a solution.
  3. In 1637, Descartes produced a geometric solution to the quadratic equation. This was the beginning of what is known as analytic geometry.
  4. Eventually the quadratic formula for the general solution to $ax^2 + bx + c = 0$ was found to be that x = [-b ± the square root of (b^2 - 4ac)] ÷ 2a.
  5. Investigation of the quadratic solution continued throughout the seventeenth and eighteenth and into the nineteenth centuries. In 1851, James Sylvester gave the name "discriminant" to $b^2 - 4ac$.
  6. If $b^2 - 4ac$ is positive, then there are two real answers for x because the discriminant has a real square root.
  7. If $b^2 - 4ac$ is 0, then there is one answer for x, which is -b/2a, because the square root of 0 is 0.
  8. If $b^2 - 4ac$ is negative, then there are two complex answers for x because the square root of a negative is imaginary.

II. Using the quadratic formula, as well as other analytical information, the graph of a quadratic function can be constructed.

- **A.** Given $y = ax^2 + bx + c$, there is a simple step-by-step method to sketching a graph of the function.
  1. The graph of a quadratic function is a parabola. As x increases in both the positive and the negative directions, the squared term becomes dominant. If a is positive, the graph will begin to increase at an increasing rate. If a is negative, the graph will decrease at an increasing rate.
  2. First, we compute the x coordinate of the vertex or turning point. The formula for the x coordinate is x = -b/2a.
  3. The x coordinate for the vertex is substituted for x in the equation and the y coordinate of the vertex is found. The vertex is now plotted.
  4. The y-intercept is the value of y when x is 0. For the quadratic, the y-intercept will always be at c.
  5. Because the parabola is symmetric, every point on the graph with the exception of the vertex has a mirror point on the other side of a vertical line that passes through the vertex.
  6. We can now plot the y-intercept and its mirror point.
  7. The x-intercept is the value of x that makes y 0. Since $y = ax^2 + bx + c$, we solve $ax^2 + bx + c = 0$ to find the x-intercepts or zeros. Each x-intercept is the mirror point of the other.
  8. If the discriminant is 0, then the vertex will be the x-intercept. If the discriminant is negative, then there is no x-intercept.
  9. If needed, additional points may be plotted for various values of x.

B. As an example, we will graph the parabola that models an object that is thrown up into the air and falls back to earth because of gravity.
   1. The equation for this model is $y = -5x^2 + 40x$, where x is the time in seconds, y is the height above the ground in meters, the -5 is one half the gravitational acceleration, and 40 is the initial velocity of the object when it was launched.
   2. The x coordinate of the vertex is $-40/-10 = 4$ and the y coordinate is $-80 + 160 = 80$ meters above the ground.
   3. Because the lead coefficient is negative, the parabola is facing down and the vertex is the maximum point. The object reaches a maximum altitude of 80 meters after 4 seconds of flight.
   4. The y intercept is 0 because the object is at ground level before it is launched.
   5. The mirror point for the y-intercept is at $x = 8$. This means that 8 seconds after the object is launched it will be back at ground level.
   6. The x intercepts are the same points as the y-intercept and its mirror point. They are the times (0 and 8 seconds) when the object is at ground level.
   7. This model is an example of a ballistic path, which is the quadratic model of a body with an initial velocity and with only the force of gravity acting on it.

C. Another form of the quadratic function is $y = a(x - h)^2 + k$, where (h, k) is the location of the vertex.
   1. An interesting example of the use of this form is to establish a model for the population density of the United States over time. The population density is y and the year of the census is x.
   2. We can note three times when density decreased as a result of the acquisition of vast amounts of territory: the Louisiana Purchase, annexation of Texas and land added after the Mexican War, and the addition of Alaska to the Union.
   3. A view of the data confirms that a quadratic model would be a reasonable choice. The vertex is chosen to be at (1790, 4.5), because the first census was in 1790 and the population density of the United States in that year was 4.5.
   4. The lead coefficient a is adjusted until the quadratic model appears to be a good fit for the data.

III. Another useful polynomial function is the cubic $y = ax^3 + bx^2 + cx + d$.
   A. The solution for the general cubic equation $ax^3 + bx^2 + cx + d = 0$ was sought for thousands of years.
      1. The first known attempt to solve the cubic was done by Babylonian mathematicians before 1600 B.C.
      2. Omar Khayyam studied geometric solutions to cubic equations in 1100 A.D.

      3. In 1545, Cardano demonstrated a solution to a special form of the cubic, $x^3 + bx = c$.
      4. Credit for the discovery of the general solution of the cubic equation (in 1832) is given to Evariste Galois, a brilliant French mathematician who died tragically at the age of 21in a duel that had political and romantic overtones.
   B. There are numerous real-world examples in which a cubic function is an appropriate model. We will look at a set of data that displays the recent growth in the number of high school soccer players in the United States.
      1. The data show the number of high school soccer players each year from 1981 through 1994. It is apparent that the number of players continued to increase during that period.
      2. What makes the data set a candidate for a cubic function is that before 1988, the number of players was increasing but at a decreasing rate. After 1988, the increase proceeded at an increasing rate.
      3. The point at which the rate of change in the rate of change (i.e., the change in the slope) goes from increasing to decreasing or from decreasing to increasing is called an "inflection point."
      4. The truly interesting question is: What happened in 1988 to make this change occur? When asked about this, many people mention the World Cup or Pele, but the actual cause has nothing to do with soccer.
      5. The application of Title IX to high school athletics caused schools with boys' soccer teams to have to offer girls' soccer.
      6. The creation of girls' teams accelerated the number of high school students who were playing interscholastic soccer.

Essential Reading:

NCTM, *Historical Topics for the Mathematics Classroom*, Sections III and V.

Supplementary Reading:

E. T. Bell, *Men of Mathematics*.

Gail Burrill, et al., *Data Analysis and Statistics across the Curriculum*.

William Dunham, *The Mathematical Universe*.

Questions to Consider:

1. Obtain timeline data on the growth of the federal debt, the growth of Medicare and Social Security payments, or the growth of consumer credit card debt. Find a quadratic function to model your data.
2. Do some research on Evariste Galois. Was he set up in a compromising situation because of his politics or was he simply a fool for love?

Lecture Nine

The Power of Exponentials

Scope: Just as humans recognized that repeated addition could be simplified by creating multiplication, repeated multiplication was simplified through the use of the exponent or power. This lecture will trace the development of the exponential notation that is in common use today. We will discover patterns as we look at the powers of the various digits. The rules for the arithmetic of numbers in exponential form will be developed logically. Applications of the exponent, such as compound interest, exponential growth, and scientific notation, will be investigated. Finally, the meaning of logarithms will be discussed.

Outline

I. As the use of mathematics became more complex, there was a need to express repeated multiplication in a more efficient manner.

  A. The trend in European mathematics during the sixteenth century was to make the expression of mathematical thought more concise.

    1. A notation was needed to allow an operation such as 5 x 5 x 5 x 5 to be written more efficiently. More important, with the development of algebra, a need existed to be able to write the product of an unknown quantity multiplied by itself.
    2. An early attempt at a new notation was accomplished by Rafael Bombelli in 1552. His method involved writing the coefficient of the variable and writing an arc above that number. In the arc would be written the power of the unknown.
    3. Using Bombelli's method, three times the square of an unknown quantity ($3x^2$ in our current notation) would be shown with a 2 in an arc written above a 3.
    4. Today we would write $5x^4$ while Bombelli would write a 4 in an arc above a 5.
    5. A modification to Bombelli's notation was made in 1619 by Jobst Burgi. He replaced the number in the arc with the use of lower-case Roman numerals.
    6. Burgi would write $3x^2$ as a 3 with ii immediately above the 3. His $5x^4$ would be a 5 with iv written above it.

  B. The seventeenth century brought the exponential notation to where it is today.

    1. In 1637, Rene Descartes wrote $3x^2$ as 3xx and $5x^4$ as 5xxxx.
    2. In 1676, Isaac Newton modified the notation of Descartes so that the power was written in the form that is still being used. The only exception was the second power, which Newton chose to write as xx in lieu of x^2.
    3. Eventually the common use of quadratic functions caused the second power to be written as it is today.
    4. Today, exponential notation appears to be very logical, yet it took over one hundred years to develop that notation.

II. When one examines the powers of the ten digits, the patterns that are observed can be divided into four types.

  A. The two most important numbers in our number system provide a very boring yet important pattern.

    1. All the powers of 0 are 0 so that a sequence of the powers of 0 would simply be 0, 0, 0, 0, …
    2. This means that if a number that ends in 0 is raised to a power, the answer will end in 0.
    3. The powers of 1are all 1. The sequence of the powers of 1 would be 1, 1, 1, …
    4. This means that if a number that ends in 1 is raised to a power, the answer will end in 1.

  B. The powers of 5 and 6 share a common pattern.

    1. The first six powers of 5 are 5, 25, 125, 625, 3125, and 15,625.
    2. The powers of 5 always end in 5. Starting with the second power, they end with 25. Starting with the third power, the digit in the hundreds place alternates between 1 and 6.
    3. If a number that ends in 5 is raised to a power, the answer will end in 5.
    4. The first six powers of 6 are 6, 36, 216, 1296, 7776, and 46,656.
    5. The powers of 6 must end in 6. Starting with the second power, there is a pattern in the tens place: 3, 1, 9, 7, 5, 3, 1, 9, …
    6. If a number that ends in 6 is raised to a power, the answer will end in 6.

  C. Similar patterns exist when powers of 2 are compared to powers of 8 and when powers of 3 are compared to powers of 7.

    1. The first eight powers of 2 are 2, 4, 8, 16, 32, 64, 128, 256. The ones place digits form a sequence of even digits: 2, 4, 8, 6, 2,…
    2. The first eight powers of 8 are 8; 64; 512; 4096; 32,768; 262,144; 2,097,152; and 16,777,216. The ones place digits form a sequence of even digits that is the reverse of the sequence for powers of 2.
    3. Note that the only even digit missing from the sequences is 0. For a number to end in 0, it must be divisible by 2 AND 5. No power of 2 or 8 is divisible by 5.
    4. The first eight powers of 3 are 3, 9, 27, 81, 243, 729, 2187, 6561. The ones place digits form a sequence of odd digits: 3, 9, 7, 1, 3,…

5. The first eight powers of 7 are 7; 49; 343; 2401; 16,807; 117,649; 823,543; and 5,764,801. The ones place digits form a sequence of odd digits that is the reverse of the sequence for powers of 3.
6. Note that the only odd digit missing from the sequences is 5. For a number to end in 5, it must be a multiple of 5 and no power of 3 or 7 is a multiple of 5.

D. The powers of 4 and 9 have patterns that represent a subset of the patterns for 2 and 3, respectively.
1. The powers of 4 are every other power of 2. The sequence of the ones place digits is 4, 6, 4, 6, …
2. The powers of 9 are every other power of 3. The sequence of the ones place digits is 9, 1, 9, 1, …

III. The rules for operating with numbers in exponential notation are not arbitrary. They are based on a logical development.

A. The meaning of raising a base to an exponent is the basis for the core operating rules.
1. Because $3^2 \times 3^5 = (3 \times 3) \times (3 \times 3 \times 3 \times 3 \times 3)$, the answer must be 3^7.
2. When multiplying numbers with the same base, keep the base and add the exponents.
3. Since $4^6 \div 4^2 = (4 \times 4 \times 4 \times 4 \times 4 \times 4) \div (4 \times 4)$, the answer must be 4^4.
4. When dividing numbers with the same base, keep the base and subtract the exponents.
5. Since $(10^2)^3 = 10^2 \times 10^2 \times 10^2$, the answer must be 10^6.
6. When raising a base to a power to a power, keep the base and multiply the powers.

B. The core operating rules are used to develop the more abstract operating rules.
1. Using the rule for multiplication, $8^2 \times 8^0 = 8^2$. Because the only number that has no effect when used in multiplication is 1, any base to the 0 power must be 1.
2. Using the rule for dividing, $9^7 \div 9^7 = 9^0$. Because a number divided by itself must be 1, any base to the 0 power is 1.
3. Using the rule for multiplication, $6^2 \times 6^{-2} = 6^0 = 1$. Therefore, the negative power must be a reciprocal.
4. Using the rule for division, $11^3 / 11^5 = 11^{-2}$. This confirms that the negative power is the reciprocal. Newton was the first person to use the negative power notation.
5. Using the rule for multiplication, $11^{1/2} \times 11^{1/2} = 11^1 = 11$. Therefore the 1/2 power represents the square root (• 11).
6. A fractional power such as a/b represents the bth root of the ath power of the number. Newton was also the first person to use the fractional power notation.

IV. There are many applications of exponential notation. Exponential growth and decay are very common, and scientific notation is used to write numbers of great magnitude, as well as exceptionally small numbers.

A. One common example of exponential growth is compound interest.
1. If \$1000 is invested to earn 8% interest, compounded quarterly, the use of an exponent helps compute the earnings from the investment.
2. Every three months the investment will pay 2% of the current balance as interest, because 2% is one-fourth of the annual interest rate.
3. At the end of the first interest period (three months), the investment pays \$20, which is .02 x \$1000. The new balance in the investment is 1.02 x \$1000 = \$1020.
4. After six months, the balance is (1.02) x (1.02) x \$1000 = \$1040.40.
5. After one year, the balance will be (1.02) x (1.02) x (1.02) x (1.02) x \$1000. It is convenient to write this in exponential form.
6. After t quarters the value of the investment will be $(1.02)^t \times \$1000$.
7. Using exponential notation, it is easy to compute the value of the investment over a significant period of time.
8. After ten years, the balance will be \$2200.04. After twenty years, the balance will be \$4875.44. After forty years, the balance will be \$23,769.91.

B. Exponential growth and decay are easily modeled with an exponential function.
1. The standard form of an exponential function is $y = ab^x$. Since y = a when x is 0, a is the y-intercept.
2. If b is greater than 1, b - 1 is the growth rate. If b is a proper fraction, then b - 1 is a negative number and represents the rate of decay.
3. A good example of exponential growth is the expansion of federal expenditures on Social Security, Medicare, and Medicaid. The exponent for a reasonable model is about 1.14, which means that the expenditures appear to be growing at an annual rate of 14%.
4. A common use of exponential decay is half life—the time it takes for one-half the mass of a radioactive substance to lose its radioactivity.
5. The exponential model for half life would be $y = a(1/2)^x$, where a is the amount of radioactive matter at the beginning and x is the number of half life periods that have passed.

C. Scientific notation takes advantage of the fact that our place values are all powers of 10.
1. The number 824,000 can be written as 8.24×10^5.
2. The number .00032 can be written as 3.2×10^{-4}.

3. Scientific notation can be used to simplify multiplication. For example 824,000 x .00032 can be multiplied using $8.24 \times 10^5 \times 3.2 \times 10^{-4} = 26.368 \times 10^1$.
4. In scientific notation the answer is written 2.6368×10^2, which of course is 263.68.

V. Logarithms are a powerful mathematical tool that use the inverse of exponential notation.

A. If $y = \log_b(x)$ then $b^y = x$. The logarithm is the inverse operation from exponentiation.

1. If $10^5 = 100{,}000$, then $\log_{10}(100{,}000) = 5$.
2. If $2^8 = 256$, then $\log_2(256) = 8$.

B. Logarithms were developed during the seventeenth century.

1. The invention (or discovery) of logarithms is credited to John Napier in 1614.
2. In 1622, William Oughtred used logarithms to create the first slide rule.
3. In 1624, Henry Briggs realized the importance of base-ten logarithms. Because the place values in our number system are all powers of 10, these logarithms were called "common logarithms."
4. Nowadays, many calculators stress the use of the natural logarithm that is based on the natural base e (see Lecture Four).

Essential Reading:

NCTM, *Historical Topics for the Mathematics Classroom*, Section V.

Sanderson Smith, *Agnesi to Zeno*, Section 39.

Lynn Arthur Steen, *On the Shoulders of Giants*, CHANGE.

Supplementary Reading:

E. T. Bell, *Men of Mathematics*.

William Dunham, *The Mathematical Universe*.

Morris Kline, *Mathematical Thought From Ancient to Modern Times*, Volume 1.

James R. Newman, *The World of Mathematics*, Volume One.

Questions to Consider:

1. Use the exponential form of compound growth to determine how much a person would have at age sixty if he or she had invested $10,000 at age thirty in an investment that had an average gain of 15% per year. How much would that same person have to invest, at age forty, in that same investment to have an equal amount at age sixty?

2. Obtain a slide rule and instructions for its use. Investigate how the rules of exponents are used in the operation of the slide rule.

Lecture Ten

Calculus: The Derivative

Scope: As scientific understanding accelerated during the seventeenth century, a need developed for a mathematics of change. The developing science needed a way to measure change when that change was not constant. Working independently, Newton and Leibniz discovered the Calculus. This lecture investigates the concept of limit, which is the foundation of calculus, and explores the derivative. We will define the meaning of the derivative, analyze the derivatives of various common functions, and discuss applications of the derivative. Finally, we will briefly explore the differential equation and its use in the real world.

Outline

I. By the middle of the seventeenth century, there was a significant demand for a mathematics of change to be used in the analysis of new scientific discoveries.

A. A most important scientific discovery was the analysis of planetary motion by Johannes Kepler.

1. Kepler's laws required a way to measure instantaneous change for a non-linear function.
2. In 1665, Newton created what he called "fluxions," which was a method for finding the rate of change at any point on a function.
3. Newton kept his discovery a secret from all but his closest friends until a rival was about to publish a similar work.
4. In 1673, Gottfried von Leibniz published his work on calculus.
5. Leibniz's work was widely praised in Europe because of his popularity. Although Newton was cold and aloof, Leibniz was a very sociable person.
6. Newton believed that Leibniz had stolen his ideas and Newton prevented Leibniz's work from gaining credibility in Britain. Today, in the English-speaking world, Newton is considered the father of calculus.
7. The truth is that most of the common notation used in calculus was developed by Leibniz.

B. The foundation of both Newton's and Leibniz's work is the concept of limit.

1. The limit of a function of x, as x approaches a specific value, can be conceived as the value that the function appears to be approaching, even if the function never reaches that value.
2. For example, the limit of $f(x) = (x^2 - 9)/(x - 3)$ as x approaches 3 is 6, even though when x is 3 the function is undefined.

3. In concrete terms, as you get closer and closer to a value of x, the limit is where it appears you will be when you reach that value of x.

II. Using the concept of limit, we can develop the derivative and then investigate the derivative of various functions.

A. Our function is defined as $y = f(x)$ and we choose two points on the function that are very close to each other. One is at x and the other at x + h, where h is a very small number.

1. The two points on the function are the ordered pairs $(x, f(x))$ and $(x + h, f(x + h))$.
2. If we draw a line segment to connect the points, we can compute the slope of that line segment using the change in y divided by the change in x.
3. The slope is $(f(x + h) - f(x))/h$ since $(x + h) - x = h$.
4. As we move the second point closer to the first point, h approaches 0 and the slope of the segment approaches the slope of the line tangent to the function at x.
5. The limit of the slope of the segment is the slope of the tangent line, which is the derivative of the function. The derivative is a function of x.
6. The derivative is defined as the limit as h approaches 0 of $(f(x + h) - f(x))/h$.
7. Various symbols are used for the derivative. The most common symbols are $f'(x)$, y', dy/dx, and Dx.

B. We can now use the definition to find the derivative of various functions.

1. We know that the derivative of $y = mx + b$ must be m (the slope).
2. Using the definition $f'(x)$ is the limit as x --> 0 of $(mx + mh + b - mx - b)/h$. This equals m, which make sense.
3. If we look at the graph of $y = x^2$, it appears that the derivative must be negative when x is negative, 0 when x is 0, and positive when x is positive.
4. Using the definition of the derivative, we find that $f'(x) = 2x$, which matches the description we obtained from observing the graph.
5. This means that when x is 5, the slope of the tangent line is 10 and when x is -2, the slope of the tangent line is -4.
6. The derivative of $y = 1/x$ is $-1/x^2$. The derivative of $y = x^3$ is $3x^2$.

III. Since the derivative measures change and much of what we deal with in real life is changing, calculus can be applied to many situations in various aspects of life.

A. The derivative is a valuable tool in the world of finance.

1. In businesses, we can measure marginal cost, marginal revenue, and marginal profit. The term "marginal" refers to the

instantaneous change in cost, revenue, and profit as one more unit of product (or service) is produced and sold.

2. The change in the money supply or the velocity of the money supply is a derivative. It is a useful tool in analyzing the financial health of a nation.
3. The second derivative, or the derivative of the derivative function, is most useful in business. It allows management to analyze how the change in sales and/or profits is changing. It can be used to predict the future.
4. The second derivative can be used to determine when to accelerate or decelerate production, when to develop a new product or advertising campaign, and when to begin cost-cutting measures to prevent a collapse of the business.

B. The derivative is vital in the world of physics.

1. We have already mentioned the need for the derivative when examining planetary motion, and the derivative is used to find the velocity and acceleration of a moving body.
2. The velocity is simply the derivative of the function that defines the distance a body has traveled as a function of time. The acceleration is the second derivative of the distance function.
3. Physics deals with flux or changes in fields, such as gravitational, electrical, and magnetic, and the derivative is used to quantify the flux.
4. There are laws of physics that deal with the conservation of energy and the conservation of momentum. The derivative is used to measure the changes that must be balanced to uphold the physical laws.
5. In electric circuits and in electromagnetic induction, change must be measured and the derivative is the tool that is used.

C. In the social and life sciences, the derivative is needed to measure change.

1. Measuring population growth and decline can be done using the derivative.
2. Changes in environmental quantities (e.g., rates of pollutants entering the environment) can be found with the derivative.
3. Changes in crime, recidivism rates for prisoners, and police effectiveness can be measured with the derivative.

IV. The differential equation is a means of creating a model to analyze the interaction between a measurement and the change in that measurement. Predator/prey analyses and models for epidemics use differential equations.

Essential Reading:

William Dunham, *The Mathematical Universe*, Differential Calculus.

NCTM, *Historical Topics for the Mathematics Classroom*, Section VII.

Lynn Arthur Steen, *On the Shoulders of Giants*, CHANGE.

Supplementary Reading:

E. T. Bell, *Men of Mathematics.*

Morris Kline, *Mathematical Thought From Ancient to Modern Times*, Volume 2.

Sanderson Smith, *Agnesi to Zeno*, Section 39.

Questions to Consider:

1. Research the life of Isaac Newton. What are his major accomplishments other than the Calculus? Investigate his personality and the possible causes for the social problems he had.
2. Newton said that the reason he saw further than others was that he "stood on the shoulders of giants." What did he mean by that statement? Who were the giants?

Lecture Eleven

Calculus: The Integral and Power Series

Scope: The derivative provides a method to measure change. One other tool that had been sought since the Babylonians was a method to measure the cumulative effect of change. This was usually symbolized by the area under a curve. The inverse of taking a derivative (differentiation) was adapted to provide an accumulated sum. This tool we call the integral. This lecture will trace the development of the integral and its use to compute the area under a curve. The second half of the lecture is devoted to the power series. The power series is a technique that allows complicated functions to be approximated by a polynomial. Using the display of a graphing calculator, we shall see how close that approximation is.

Outline

I. For thousands of years, a method to compute areas bounded by curves had been sought without much success.

A. In the third century B.C., Archimedes discovered a method to approximate the area of an irregularly shaped surface.

1. He called his method computation by exhaustion, because he would segment areas into shapes for which he could find the area (triangles, rectangles, etc.).
2. Each step in the method of exhaustion provided smaller and smaller areas to be analyzed.
3. Unfortunately a limited number of copies of the book in which Archimedes explains this method were produced and after some time, the method was forgotten.
4. In the fourteenth century, scholars found the explanation of the method of exhaustion and work proceeded in trying to develop a version that would obtain precise areas.

B. The creation of calculus offered an opportunity to provide precise answers to measurements of area.

1. The antiderivative is the inverse of the derivative. If the derivative of $5x^2$ is $10x$, then the antiderivative of $10x$ is $5x^2$.
2. Since the derivative of a constant is 0, the antiderivative might contain a constant term, so we say that the antiderivative of $10x$ is $5x^2 + C$, where C is a generic constant.
3. Just as we would use multiplication facts to develop division facts, we use our knowledge of derivatives to evaluate antiderivatives.
4. The antiderivative is called the integral and the symbol is a stretched S (shown as •).

5. When we use the integral to find the antiderivative function, we are said to be taking the indefinite integral.
6. We can generalize from our knowledge of derivatives to state that the indefinite integral of $ax^n = a/(n+1)\ x^{(n+1)} + C$.

II. The question to be answered now is: How do we use the indefinite integral, which provides a function for an answer, to measure the area under a curve?

A. We begin by defining a function, f(x), and calling the area between the graph of y = f(x) and the x-axis, from x = a to x = b, A(x).

1. Because A is a function of x, we can talk about the derivative of A with respect to x, dA/dx.
2. The derivative dA/dx is the additional area added to A(x) if we increase x by an exceptionally small amount.
3. Examining a picture of A(x) shows that the very thin sliver of area we add to A is the function y.
4. If dA/dx = y then, taking the antiderivative, we get that A(x) is the integral of y with respect to x.
5. The integral allows us to evaluate the cumulative effect of the small changes in x from a to b by evaluating the integral at b and at a and then subtracting to find the difference.
6. If I wish to find the area under the curve of $y = x^2$ between x = 0 and x = 3, I integrate the function x^2, which gives me $x^3/3$.
7. The value of $x^3/3$ at x = 3 is 9, and its value at x = 0 is 0. The difference 9 - 0 = 9, so the area under the curve is 9.
8. You may ask why was there no + C? Because 9 + C - (0 + C) still equals 9 (because the Cs cancel), there is no reason to complicate the notation.
9. To find the area beneath the graph of $y = x^3$ from x = 1 to x = 3, we integrate the unction and get $x^4/4$. Subtracting 81/4 - 1/4 provides a calculated area of 20.

B. The definite integral method for computing the area under a curve was first developed by the Italian mathematician and Jesuit priest Francesco Bonaventura Cavalieri (1598–1647).

1. Cavalieri had been a student of Galileo.
2. He looked upon his work as being geometric.

C. The definite integral is used to define the natural logarithm.

1. The natural logarithm is the logarithm whose base is e.
2. The natural logarithm of the number a is defined as the definite integral of 1/x from x = 1 to x = a.

D. There are many useful applications of the definite integral.

1. In statistics, integration is used to find specific areas under probability curves, such as the normal distribution. The numbers in most statistical tables are obtained by integration.
2. In physics, there are many quantities that are calculated using integration. One example is work, which is the integral of the

product of the force applied multiplied by the differential distance over which the force is applied.

3. If the change in population over time is a function, then the definite integral of that function will provide the total growth or loss of population during the interval over which the integral is taken.

III. As the applications for which mathematics was used became more complicated, mathematicians were dealing with such models as exponential functions, logarithmic functions, and trigonometric functions. A process was needed to approximate these more complex functions with a simpler model.

A. The power series was developed to approximate various types of functions with a polynomial.

1. Series means a sum of a sequence of terms. The terms of a power series were monomials that each contained a power of x.
2. The power series was first developed by Brook Taylor in 1715.
3. He died at the age of 46 with his work unfinished. Colin Maclaurin continued Taylor's work and published his treatise in 1742.
4. Two common types of power series studied in calculus are named for Taylor and Maclaurin.
5. The ironic fact is that the Taylor series and the Maclaurin series were both developed by Johann Bernoulli. As has been stated in earlier lectures, the person who gets credit for a mathematical concept or algorithm may have had little or nothing to do with the creation of that which bears his or her name.

B. The Maclaurin series are the simpler, so we will first look at the Maclaurin series for the exponential e^x.

1. The Maclaurin series for e^x is $1 + x + x^2/2! + x^3/3! + \ldots$
2. The notation 3! is read "three factorial" and means 3 x 2 x 1. It should be noted that 0! is defined as 1.
3. If we used this power series to evaluate e, we would replace x with 1 since $e^1 = e$. If we used the first six terms of the series, we would have $e = 1 + 1 + 1/2 + 1/6 + 1/24 + 1/120$.
4. The sum of those six terms is 2.71666… which is quite close to 2.7182818… If we were disappointed with the precision, we could add more power series terms to refine our accuracy.
5. If we look at the graph of the exponential function and the polynomial made up of the first six terms of the power series, we can see that between x = -s and x = 2, the power series is quite close to the function.
6. If we add more terms to the power series, we improve the estimate. The key is knowing for what values is the power series a valid approximation.

C. Finally, we will examine the Maclaurin series to approximate the sine of x.

1. The Maclaurin series for the sine of x is $x - x^3/3! + x^5/5! - x^7/7! + \ldots$
2. The terms have alternating signs and only the odd powers of x are used. The value of x must be expressed in radians because the basis for the calculus of trigonometric functions requires the angles to be measured in radians (2π radians = 360 degrees).
3. If we use the first four terms of the Maclaurin series to estimate the sine of 30 degrees ($\pi/6$ radians), our result is .50000213. This is very close to the sine of 30 degrees, which is .5.

Essential Reading:

Morris Kline, *Mathematical Thought From Ancient to Modern Times*, Volume 2.

NCTM, *Historical Topics for the Mathematics Classroom*, Section VII.

Supplementary Reading:

E. T. Bell, *Men of Mathematics*.

Questions to Consider:

1. Find the Maclaurin series for y = ln (x), where ln is the natural logarithm or the logarithm base e. Use this series to find the ln (10) and check the accuracy of your estimate on a calculator.
2. Find the area of a semi-circle by the process of exhaustion using sections that are right triangles.

Lecture Twelve

Fractals

Scope: Much of the mathematics we learn in school or at the university was created hundreds or even thousands of years ago. Most modern mathematics is typically far beyond the average person. The advent of the powerful yet inexpensive computer has permitted research into a branch of mathematics that has been disregarded until the final quarter of the twentieth century. That mathematics is the investigation of fractals. Fractals present images that are incredibly beautiful and complex. The mathematics that is the foundation of fractals can be understood by most educated people. The ultimate positive factor about fractals is that they are extremely useful for as widely diverse applications as storing images and analyzing chaotic events. This lecture will provide a brief taste of what fractals have to offer.

Outline

I. The modern study of fractals began in 1976, when Benoit Mandelbrot, a French mathematician, published *Les Objets Fractals*.

A. Mandelbrot had worked for twenty-five years studying seemingly chaotic events. With the availability of computing power, he was able to analyze the data and discovered that there were discernible patterns.

1. His work began with the analysis of telephone line errors completed for Bell Laboratories.
2. He also examined the measurements of the height of the Nile River over time, the fluctuation of commodity prices, and measurements of turbulence.
3. All these data sets seemed to be records of random events. Mandelbrot was able to find patterns in the chaotic outcomes. Fractal mathematics helped define the patterns.

B. At this time, it is necessary to define what a fractal is.

1. A fractal figure has a dimension that is not a whole number.
2. A fractal image displays self-similarity. If you magnify a section of the image, you see the original image—a good example is a fern.
3. A fractal function will display seemingly chaotic behavior.

II. There was a body of work done on fractals before Mandelbrot but the lack of computer power stymied the development of this branch of mathematics.

A. The initial investigations involved the production of fractal images using a repetitive process.

1. In 1883, Georg Cantor created the Cantor set. He took the number line from 0 through 1 and removed the middle one-third.

2. The middle one-third of each of the two remaining pieces was removed and the process was repeated over and over again.
3. There are an infinite number of points within the pattern after an infinite number of repetitions of the process. Mathematicians of the day gave little importance to Cantor's set.
4. In the 1890s, Guiseppe Peano created the Peano curve, which was constructed by repeating a process. The curve appeared to be a one-dimensional object that filled the plane.
5. Around 1900, Helge von Koch created the Koch curve. He started with a line segment that was split into three equal pieces. A fourth equal piece was added and a shape similar to _/_ was formed.
6. The same technique was performed on each of the four line segments. And this process was repeated over and over again.
7. The Koch curve had infinite length despite the fact that the distance between its endpoints was finite.
8. If the process is begun using an equilateral triangle, the fractal figure is called the Koch snowflake.

B. In 1916, Waclav Sierpinski created the Sierpinski triangle, which has become an important image in fractal geometry.

1. Sierpinski started with an equilateral triangle. The midpoints of the three sides were connected, forming four smaller equilateral triangles.
2. The middle triangle is removed and the process is repeated for the three remaining triangles.
3. After a number of repetitions of the process, the figure is obviously a fractal, because magnification of any piece of the triangle produces a complete Sierpinski triangle.
4. When computers became available to investigate fractals, the Sierpinski triangle became a very common result. We can also find the Sierpinski triangle in Pascal's triangle (see Lecture Two).
5. An example of this is the chaos game. You start with an equilateral triangle with each vertex identified with a color—red, blue, green. A six-sided fair die that has two red sides, two blue sides, and two green sides is rolled.
6. The game starts by picking a random point inside or outside the triangle. The die is rolled and a new point is marked halfway between the old point and the vertex whose color was rolled. The process is repeated over and over.
7. Once the point is in the triangle, allow a few more rolls and then begin plotting points.
8. One would expect that after many repetitions, the triangle's interior would fill up with random points. What actually happens is that the points form a Sierpinski triangle.

C. At the beginning of the twentieth century, Gaston Julia and Pierre Fatou began work on iteration theory.

1. An iteration process takes the output from a process (y) and uses it as the input (x) in the next step. This process is repeated again and again.
2. Julia and Fatou believed that applying the iterative process to functions would produce images. They were correct, but they were not able to produce the images.
3. Sixty years later, the Julia set was produced and the image is universally recognized for its beauty and complexity.

D. Common mathematical functions can be used to demonstrate the iteration process.
   1. Enter a positive number in a calculator. Take the square root of the number and then take the square root of the answer. Continue to iterate and eventually the process will lead to the number 1.
   2. Enter a number in the calculator that is greater than 1. Square that number and continue to square the output in an iterative process. Eventually you will get an error message from the calculator. You have reached an overload situation. The process is leading to infinity.
   3. Start with a number between 0 and 1 and apply the iterative squaring process. Eventually, you will reach 0.
   4. When we examine iterations of the function $y = x^2 - 2$, there is an interesting and unexpected result.
   5. If our starting point is greater than 2, the process leads to an infinite result.
   6. If we start with 0, 1, or 2, we get a lock very quickly, but if we start with a number between 1 and 2, there is no apparent pattern to the process. We have chaos.
   7. Iteration of the sine and cosine functions yields interesting results.

III. Mathematicians are barely scratching the surface of the knowledge that will be discovered using fractals.
   A. As computers become faster, more investigations will be conducted.
   B. Someday, the path of a hurricane will be able to be modeled with reasonable accuracy using a fractal function.
   C. The potential is unlimited. As we view the beauty of the fractal images, it is hard to believe that we are looking at science.

Essential Reading:

Linda Garcia, *The Fractal Explorer.*

Robert L. Devaney, *Chaos, Fractals and Dynamics.*

Supplementary Reading:

American Mathematical Society, *Chaos and Fractals.*

Benoit B. Mandelbrot, *The Fractal Geometry of Nature.*

Heinz-Otto Peitgen, *Fractals for the Classroom*, Volume One.

Questions to Consider:

1. Construct a Pascal's triangle with twenty-one rows. Color in the numbers that are even and you will see a Sierpinski triangle.
2. Complete an Internet search to find real-world applications that use fractals.

Note: Images of the Mandelbrot set, created from mathematical functions and using the "complex plane" and computer power, are presented on the videotape. There are also books and even calendars showing images from the Mandlebrot set.

Timeline

2900 B.C. .... Construction of the first Egyptian pyramid

2000 B.C. .... Babylonians calculate the area of a triangle

1650 B.C. .... Egyptians show that the volume of a square-based pyramid is one-third the volume of a square-based prism of the same height

1600 B.C. .... Babylonians attempt to solve a cubic equation

1500 B.C. .... Egyptians are using sundials

600 B.C. .... Greek letters used as numerals

585 B.C. .... Thales predicts a solar eclipse

Sixth century B.C. .... Pythagorean theorem

500 B.C. .... First recorded use of Roman numerals

414 B.C. .... Squaring the circle is mentioned in the play *The Birds* (Aristophanes)

300 B.C. .... Euclid writes *The Elements*

Third century B.C. .... Sieve of Eratosthenes

225 B.C. .... Apollonius publishes his book on conic sections

Second century .... Greeks use the letter *omicron* to represent "nothing"

470 .... Chinese use 355/113 for the ratio between circumference and diameter

500 .... Indians begin the formalized study of trigonometry

825 .... Use of place value accepted in the Arab world

Thirteenth century .... Hindu-Arabic number system accepted in Europe

1260 .... Latin translation of Euclid's *Elements*

1453 .... Arab armies conquer Constantinople; Greek scholars flee to Europe

1557 .... The symbol "=" first used

1615 .... Johannes Kepler attempts to measure the volume of a cylindrical wine cask

1618 .... John Napier invents logarithms

1637 .... René Descartes introduces analytic geometry

1665 .... Isaac Newton creates fluxions; Blaise Pascal describes his triangle

1673 .... Gottfried von Leibniz publishes his work on calculus

1676 .... Isaac Newton develops the exponential notation that is still in use today

1700 .... European bookkeepers accept the use of Hindu-Arabic numerals

1704 .... Edmond Halley studies the path of a comet

1715 .... Brook Taylor develops the first power series

1718 .... Abrahan De Moivre publishes *Doctrine of Chance*

1730 .... Abraham De Moivre produces a method to find the complex roots of a real number

1736 .... Leonhard Euler uses π for the ratio between circumference and diameter of a circle

1742 .... Colin Maclaurin publishes his treatise on power series

1810 .... Pierre Simon Laplace starts his work on the Central Limit theorem

1821 .... Augustin Louis Cauchy is the first to use the term "conjugate"

1831 .... Carl Gauss coins the term "complex number"

1832 .... Evariste Galois produces the general solution of the cubic equation

1857 .... Florence Nightingale analyzes British mortality rates in the Crimean War

1883 .... Georg Cantor creates the Cantor set

1900	Helge von Koch creates the Koch curve; Karl Pearson invents the chi square statistic
1916	Waclav Sierpinski creates the Sierpinski triangle
1936	George Gallup uses quota sampling to correctly predict the results of the 1936 presidential election
1948	Sampling techniques prove inadequate in predicting the 1948 presidential election
1974	The discovery of "Hank Aaron numbers"
1976	Benoit Mandelbrot publishes *Les Objets Fractals*
2002	The second palindrome year in eleven years (the other was 1991)

Glossary

Bernoulli trial (18): a series of events in which each event has only two possible outcomes and each event is independent of the others.

binomial distribution (18): a distribution of Bernoulli trials in which the number of events in each trial is fixed.

box plot (20): a graphical display based on the minimum, lower quartile, median, upper quartile, and maximum values of a set of data.

Cartesian plane (13): the coordinate plane formed by two perpendicular number lines.

Central Limit theorem (18): a definition of the shape, expected value, and variation of a sampling distribution.

coefficient (6): a constant term in a monomial; 3 is the coefficient in 3x.

complement (17): in probability, the complement of "winning" is "not winning."

complex number (5): a + bi, where a and b are real numbers and i is the square root of -1.

confidence interval (23): an interval estimate for a population parameter based on a sample statistic.

conjugate (5): the conjugate of a + bi is a - bi.

correlation (21): a measure of the relationship between two variables.

cosecant (15): the ratio of the hypotenuse to the opposite side in a right triangle.

cosine (15): the ratio of the adjacent side to the hypotenuse in a right triangle.

cotangent (15): the ratio of the adjacent side to the opposite in a right triangle.

cubic equation (8): a polynomial equation in which the highest power of the variable is 3.

definite integral (11): an integration technique designed to compute the area between a curve and the x-axis.

derivative (10): a function that defines the slope of the tangent line at any point on a given function.

e (4): the natural base.

ellipse (16): a two-dimensional figure in which every point satisfies the requirement that the sum of the distances from the point to the two foci is constant.

exploratory data analysis (20): the qualitative analysis of data.

exponential decay (9): $y = ab^x$, where b is a number between 0 and 1.

exponential equation (9): an equation of the form $y = ab^x$.

Fibonacci number (2): a number in the sequence 1, 1, 2, 3, 5, 8, 13, 21, 34…

fractal (12): a self-similar geometric figure.

geometric distribution (19): a distribution of Bernoulli trials in which each trial lasts until the first success is obtained.

Hank Aaron numbers (2): a pair of consecutive numbers whose product is the product of the first n prime numbers.

histogram (20): a graphical display that uses bars to indicate the frequency of occurrence of data in various classes.

hyperbola (16): a two-dimensional figure in which every point satisfies the requirement that the difference between the distances from the point to the two foci is constant.

hypothesis test (22): a statistical test designed to assess significant change.

imaginary number (5): a number whose square is negative.

integer (3): a positive or negative whole number, or zero.

integral (11): the inverse operation of the derivative.

interquartile range (20): the distance between the upper and lower quartiles of a set of data.

irrational number (4): a number that cannot be written as the ratio between two integers.

iteration (12): a repetitive process in which the output of one step is used as the input for the next step.

line plot (20): a graphical display of data.

linear equation (7): an equation of the form $y = mx + b$, the graph of which is a line.

level of significance (22): the probability that the difference between the results of a sample and the expectations of the null hypothesis was due to chance variation.

margin of error (23): the distance in a confidence interval from the sample statistic to the one edge of the interval.

modulo system (1): a system that has a limit on the numbers used (for example, the modulo 6 system uses only 0, 1, 2, 3, 4, 5).

negative binomial distribution (19): a distribution of Bernoulli trials in which each trial lasts until a fixed number of successes is obtained.

normal distribution (18): a symmetric exponential function that is an extremely common distribution (also known as the bell curve).

null hypothesis (22): the statement that no significant change will result from a treatment or selection process.

outlier (20): a data point significantly far away from the bulk of the data.

palindrome (2): a word or number that is the same forward or backwards, such as "radar" or 13931.

parabola (16): a two-dimensional figure in which every point satisfies the requirement that the distance from the point to the focus is equal to the perpendicular distance from the point to a line called the directrix.

pi (4): the ratio of the circumference of a circle to its diameter.

Pascal's triangle (2): a sequence of rows of numbers with many useful mathematical applications.

polynomial (6): the sum of one or more terms of the form ax^n, where a is a real number and n is a whole number.

postulate (13): a statement accepted as true without proof.

power series (11): a polynomial used to approximate a function that is not a polynomial.

Pythagorean theorem (13): for any right triangle, the sum of the squares of the two legs equals the square of the hypotenuse.

quadratic equation (8): a polynomial equation in which the highest power of the variable is 2.

rational number (3): a number that can be written as the ratio of two integers.

regression (21): a process to identify a function to model a set of bivariate data.

residual (21): the difference between the actual value of the response variable for a data point and value predicted by the regression model.

secant (15): the ratio of the hypotenuse to the adjacent side in a right triangle.

sine (15): the ratio of the opposite side to the hypotenuse in a right triangle.

slope (7): the amount of change of y , in a linear function, if x increases by 1.

squaring the circle (14): finding a square whose area is equal to the area of a given circle.

standard error (22): a theoretically obtained standard deviation.

tangent (15): the ratio of the opposite side to the adjacent in a right triangle.

Biographical Notes

Apollonius (c. 225 B.C.). Author of *Conic Sections*, a monumental text on the investigation of ellipses, parabolas, and hyperbolas, terms that are used to label the various curves. He was given the title "the Great Geometer" by his peers.

Archimedes (287–212 B.C.). The greatest mathematician of antiquity. His work on finding area and volume through the method of exhaustion was a precursor of integral calculus. He was most proud of his work with spheres and cylinders. He was also known for his mechanical inventions, many of which had a military purpose.

Geralamo Cardano (1501–1576). He was responsible for the beginning of a mathematical analysis of probability, which he developed through successful gambling. Cardano also worked on solutions to cubic and quartic (fourth-degree) equations.

Abraham De Moivre (1667–1754). His book *Doctrine of Chance* (1718) was an important treatise on probability. His work with complex numbers provided a means to find the complex roots of a real number (for example, all three cube roots of the number 1, of which two are complex).

Rene Descartes (1596–1650). Renowned philosopher and mathematician. He is credited with "I think, therefore I am." His creation of the Cartesian plane (graph paper) and analytic geometry paved the way for the invention of calculus.

Euclid (c. 300 B.C.). The "father of Geometry." His *Elements* is the fundamental text for geometry. He also proved that the Pythagoreans were incorrect in believing that all numbers were rational.

Leonhard Euler (1701–1783). The most prolific writer on mathematics in history. He is credited with labeling e (the natural base), i (the square root of - 1), and Π (the universal ratio between a circle's circumference and its diameter).

R. A. Fisher (1890–1962). Working with agricultural and genetic experiments, he created the method for the design and evaluation of statistical experiments. He is also pioneered the use of randomization in experiments.

Evariste Galois (1811–1832). A brilliant mathematician whose life was ended prematurely in a duel. He was the first to provide a general solution for the cubic equation. Much of his work was not understood until years after his death.

Carl Gauss (1777–1855). He was known to have been a brilliant child and he devoted his life to work in many areas of mathematics. His work with the normal probability distribution has caused many to refer to the graph of that distribution as the Gaussian curve.

Omar Khayyam (1048–1131). This Persian is best known for his poetry, but he did critical work in geometry and the solution of algebraic equations, as well. It is believed that he worked with what we now call Pascal's triangle.

Gottfried von Leibniz (1646–1716). Although the English-speaking world generally credits Isaac Newton with discovering calculus, Leibniz discovered calculus independent of Newton. His notation was universally adopted and is still used in calculus today.

Leonardo of Pisa (1170–1250). He is also known as Fibonacci, the name given to the sequence that appeared in his book *Liber abacci* . He was the premier European mathematician of the thirteenth century and was responsible for the European acceptance of the Hindu-Arabic number system.

Benoit Mandelbrot (1924–). His work *Les Objets Fractals* began the modern study of fractals, self-similar curves that are being researched to better understand chaos.

Isaac Newton (1642–1727). A most brilliant mathematician and scientist. He discovered calculus and developed the laws of mechanics, light, and heat. Everyone knows the story of Newton seeing a falling apple and realizing that the laws that governed its fall also govern the movement of the moon around the earth. His version of calculus, which he called fluxions, was created in response to the need for mathematics to deal with the latest theories in astronomy.

Blaise Pascal (1623–1662). The numerical triangle that bears his name was well known many years before his publication *Traite du triangle arithmetique*. His work in probability before his premature death heightened the interest in the subject during the seventeenth century.

Karl Pearson (1857–1936). Invented the chi square statistic, the oldest inference method still in use today, in 1900. His work in statistics helped bring about the understanding that statistics was a discipline separate from mathematics.

Pythagoras (572–497 B.C.). He founded a movement, the Pythagoreans, who did significant work in the structure of numbers and the mathematics of music. Although the fact that the sum of the squares of the two legs of a right triangle is equal to the square of the triangle's hypotenuse was known before the sixth century B.C., the Pythagoreans are given credit for the theorem.

Thales (c. 580 B.C.). One of the early contributors to the golden age of Greek mathematics, Thales was responsible for the development of the geometric proof. He was also known as a prolific problem solver. Unfortunately, none of his original works has survived.